STONE-CECH REMAINDERS WHICH MAKE CONTINUOUS IMAGES NORMAL

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ABSTRACT. If f is a continuous surjection from a normal space X onto a regular space Y , then there are a space Z and a perfect map $bf : Z \rightarrow Y$ extending f such that $X \subset Z \subset \beta X$. If f is a continuous surjection from normal X onto Tychonov Y and $\beta X \setminus X$ is sequential, then Y is normal. More generally, if f is a continuous surjection from normal X onto regular Y and $\beta X \setminus X$ has the property that countably compact subsets are closed (this property is called C -closed), then Y is normal. There is an example of a normal space X such that $\beta X \setminus X$ is C -closed but not sequential. If X is normal and $\beta X \setminus X$ is first countable, then $\beta X \setminus X$ is locally compact.

We began the study of the class of spaces ACRIN (all continuous regular images normal) in [FL], where we showed that if X is a normal space such that $\beta X \setminus X$ is finite, then X is ACRIN. Here, we give more general conditions on $\beta X \setminus X$ which imply that a normal space is ACRIN.

For a Tychonov space X , we denote the Stone-Cech compactification of X by βX . We call a space Z such that $X \subset Z \subset \beta X$ an *intermediate space*. If X and Y are Tychonov spaces and $f : X \rightarrow Y$ is a continuous map, we denote by $\beta f : \beta X \rightarrow \beta Y$ the Stone extension of f .

Let us recall some well-known properties of maps.

- 1. Lemma.** *Let $f : X \rightarrow Y$ be a continuous surjection. (a) If X is normal and f is a closed mapping, then Y is normal.*
(b) If X is countably compact, then Y is countably compact.
(c) If f is perfect and Y is countably compact, then X is countably compact.
(d) If $\beta f^\rightarrow(\beta X \setminus X) = \beta Y \setminus Y$, then f is perfect.

Let us fix a continuous surjection $f : X \rightarrow Y$ from a normal space to a regular space. Let $bf : P \rightarrow Y$ be a perfect map extending f , where P is an intermediate space. If Y is Tychonov, we will set $P = \beta f^\rightarrow(Y)$ and

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$bf = \beta f|P$. We defer the discussion of how to define P and bf in the case when Y is regular but not Tychonov. From Lemma 1, we see that if P is normal, then Y is normal. Thus, we seek conditions which imply that P is normal.

One such condition is to assert that every intermediate space Z is normal. Barr and Hajek introduced this notion in [BH] and called it normality inducing. Further, they showed that if X is normality inducing, then X is countably compact, and they showed that X is normality inducing if and only if every compact subset of $\beta X \setminus X$ is finite.

A condition which implies that an intermediate space Z is normal is that $Z \setminus X$ is closed in $\beta X \setminus X$. Then Z is normal because it is the union of the normal space X and the compact space $\text{Cl}_{\beta X}(Z \setminus X)$. (See [FL, Lemma 1.1(c)].) Thus, if X is normal and $\beta X \setminus X$ is finite, or more generally, discrete, then X is normality inducing, and hence ACRIN.

Let us assume that X is countably compact. Again from Lemma 1, we see that P is countably compact. Thus, it is not necessary that every intermediate space Z be normal; it is enough to require that countably compact intermediate spaces be normal.

2. Proposition. *If $\beta X \setminus X$ is sequential and Y is Tychonov, then $P \setminus X$ is closed in $\beta X \setminus X$. Hence, Y is normal.*

Proof. Suppose that $q \in \beta X \setminus X$ and there is a sequence $(p_n)_{n \in \omega}$ in $P \setminus X$ converging to q . By the definition of P , each $\beta f(p_n)$ is an element of Y . Since Y is countably compact, $(\beta f(p_n))_{n \in \omega}$ has a cluster point y in Y . By continuity, $(\beta f(p_n))_{n \in \omega}$ converges to y and $y = \beta f(q)$. Therefore, $q \in P$, and $P \setminus X$ is sequentially closed in $\beta X \setminus X$. Because $\beta X \setminus X$ is sequential, $P \setminus X$ is closed in $\beta X \setminus X$. $\square$

A space is called *C-closed* if every countably compact subset is closed. (See [IN].) For example, sequential spaces are *C-closed*. Countable spaces, which need not be sequential, and *P-spaces*, which are sequential if and only if they are discrete, are *C-closed*. If X is normal and not countably compact, then X contains a closed copy of ω . Thus, $\beta X \setminus X$ contains a closed copy of $\beta\omega \setminus \omega$. Therefore, $\beta X \setminus X$ is not *C-closed*—it contains the nonclosed countably compact subset $(\beta\omega \setminus \omega) \setminus \{p\}$ where $p \in \beta\omega \setminus \omega$.

3. Theorem. *If $\beta X \setminus X$ is C-closed, then every countably compact intermediate space Z is normal. Hence X is ACRIN.*

Proof. Towards a contradiction, assume that H and K are disjoint closed subsets of a countably compact intermediate space Z and that $q \in [\text{Cl}_{\beta X}(H) \cap \text{Cl}_{\beta X}(K)] \setminus Z$. If q were an element of $\text{Cl}_{\beta X}(H \cap X) \cap \text{Cl}_{\beta X}(K \cap X)$, then X would not be normal. Assume without loss of generality that $q \notin \text{Cl}_{\beta X}(H \cap X)$. By regularity, there is an open subset U of βX with $\text{Cl}_{\beta X}(H \cap X) \subseteq U \subseteq \text{Cl}_{\beta X}(U) \subseteq \beta X \setminus \{q\}$. Then $H \setminus U$ is a countably compact subset of $\beta X \setminus X$

(since it is a closed subset of Z), and since $\beta X \setminus X$ is assumed to be C -closed, $q \in \text{Cl}_{\beta X \setminus X}(H \setminus X) = H \setminus U \subseteq Z$, contradicting $q \notin Z$. $\square$

If Y is a locally compact space and $\alpha Y = Y \cup \{\infty\}$ is its one-point compactification, then for large enough cardinal γ , the subspace $(\alpha Y \times \gamma) \cup (\infty, \gamma)$ of $\alpha Y \times (\gamma + 1)$ is a normal space whose Stone-Cech remainder is Y . Thus, every locally compact space is a remainder of a normal space. The following result limits the applicability of Theorem 3 to exactly the locally compact spaces if $\beta X \setminus X$ is first countable.

4. Proposition. *If X is normal and $\beta X \setminus X$ is first countable, then $\beta X \setminus X$ is locally compact.*

Proof. Towards a contradiction, suppose that $p \in \beta X \setminus X$ has a countable nested base in $\beta X \setminus X$ such that for all n , $\text{Cl}_{\beta X \setminus X}(B_n)$ is not compact. Since $\text{Cl}_{\beta X}(B_n)$ is compact, we may choose distinct $x_n \in \text{Cl}_{\beta X}(B_n) \cap X$. Then $\{x_{2n} : n \in \omega\}$ and $\{x_{2n+1} : n \in \omega\}$ are disjoint closed subsets of X , both of whose closures in βX contain p . This contradicts the normality of X . $\square$

There are normal spaces whose Stone-Cech compactifications are C -closed, but not locally compact. For example, it is not hard to show that if E is an η_1 -set with the order topology and X is the set of non- P -points of the Dedekind compactification of E , then $\beta X \setminus X$ is a P -space without isolated points, and hence is C -closed but not locally compact. Since X , a linearly ordered space is normal, it follows from Theorem 3 that every continuous image of X is normal.

We now give alternative definitions of bf and P which require only that Y be regular. The general situation is this: f is a continuous surjection from the normal space X to a regular space Y , and Z is a Hausdorff extension of X , that is, a Hausdorff space which contains X as a dense subspace. For $p \in Z$, define $\mathcal{N}_p = \{N \cap X : N \text{ is a neighborhood of } p \text{ in } Z\}$. Let G be $\text{Cl}_{Z \times Y}(\text{graph } f)$.

5. Lemma. *Let f , X , Y , Z , and G be as above. (a) For $p \in Z \setminus X$, $f \cup \{(p, y)\}$ is continuous if and only if $f[\mathcal{N}_p]$ converges to y .*

(b) If for all $p \in Z$, $f \cup \{(p, y_p)\}$ is a continuous function, then $f \cup \{(p, y_p) : p \in Z\}$ is a continuous function.

(c) $(p, y) \in G$ if and only if $f[\mathcal{N}_p]$ adheres to y .

Further assume that X is normal and Z is βX .

(d) If $f[\mathcal{N}_p]$ adheres to y , then $f[\mathcal{N}_p]$ converges to y .

(e) Hence, G is the graph of a function bf and bf is perfect.

Proof. (a) and (b) are [PW, 4.1(1) and 4.1(n)]; (c) is routine. Proof of (d): Because X is normal and $Z = \beta X$, we may consider points of Z to be ultrafilters of closed subsets of X and basic open sets have the form $N(F) = \{q \in \beta X : F \notin q\}$. We prove the contrapositive. Suppose that $f[\mathcal{N}_p]$ does not converge to y . There is a neighborhood V of y which does not contain $f^\rightarrow(N)$

for any $N \in \mathcal{N}_p$. Because Y is regular, there is an open W such that $y \in W$ and $\text{Cl}_Y W \subseteq V$. Consider $H = f^{\leftarrow}(\text{Cl}_Y W) = \{x \in X : f(x) \in \text{Cl}_Y W\}$. Then $N(H) \in \mathcal{N}_p$ and $f^{\rightarrow}(N(H)) \cap W = \emptyset$. Thus, $f[\mathcal{N}_p]$ does not adhere to y .

We have arranged things so that the proof of (e) is easy. By (c), (d), and the fact that points of convergence are unique in Hausdorff spaces, bf is a function. Continuity follows from the previous parts. Because Z is compact, the projection onto Y is a closed map; bf is the restriction of projection to the closed set G , so it is also closed. Finally, $bf^{\leftarrow}\{y\}$ is $G \cap (Z \times \{y\})$, the intersection of a closed set and a compact set. $\square$

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We close with some questions.

Question 1. If X is normal and $\beta X \setminus X$ is countable, is $\beta X \setminus X$ sequential? If X is normal and $\beta X \setminus X$ is sequential, is $\beta X \setminus X$ locally compact?

Question 2. If X is normal and $\beta X \setminus X$ has countable tightness, is X ACRIN? Is there, without extra axioms of set theory, a regular space of countable tightness which is not C -closed? (Balogh proved, assuming PFA, that locally compact spaces of countable tightness are C -closed. Fedorchuk constructed, assuming $\diamond$, a Tychonov space of countable tightness which is not C -closed.)

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