WHEN ARE REES CONGRUENCES PRINCIPAL?

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ABSTRACT. Let ρ_I be the Rees congruence modulo the ideal I of the free monoid X^* . In this short note we give necessary and sufficient conditions, in terms of the partial order induced by division on the complement of I , for ρ_I to be principal. In particular, we prove that if I is principal, so is ρ_I .

INTRODUCTION

Let X be a finite alphabet with at least two letters and X^* the free monoid on X . Given a language L on X , the *principal congruence* determined by L , and denoted P_L , is defined by: $u \equiv v(P_L)$ if the following is valid: for all $x, y \in X^*$, $xuy \in L$ if and only if $xvy \in L$. Recall that L is a union of P_L -classes and that P_L is the coarsest congruence saturating L . A congruence ρ on X^* is said to be *principal* if $\rho = P_L$ for some $L \subseteq X^*$. If I is an ideal of X^* , the Rees congruence ρ_I is defined by: $u \equiv v(\rho_I)$ if either $u, v \in I$ or $u = v$ when $u, v \notin I$.

In this short note we find necessary and sufficient conditions for the Rees congruence ρ_I to be principal and deduce that the Rees congruence modulo a prime ideal is principal as is the Rees congruence modulo a principal ideal.

As for notation, the length of an element $w \in X^*$ (called a *word*) will be denoted by $|w|$ and the complement of a language L over X will be denoted by $\bar{L}$.

As a general reference we recommend G. Lallement's book [2].

MAIN RESULTS

Definition 1. (a) Define a partial order $\leq$ on X^* by $u \leq v$ if $v = puq$ for some $p, q \in X^*$.

(b) A set of incomparable elements relative to the partial order in (a) is called an *infix code* [3].

Notation. If L is a language over X , the sets $\{u \in X^* | u \leq l \text{ for some } l \in L\}$ and $\{u \in X^* | l \leq u \text{ for some } l \in L\}$ will be denoted by (L) and $[L)$, respectively.

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Lemma 2. Let I be an ideal of X^* and L a language over X . If $L \subseteq I$, then $[L] \subseteq I$ while if $L \subseteq \bar{I}$, the complement of I , then $[L] \subseteq \bar{I}$.

Proof. Clear from the definitions. $\square$

Definition 3. A language $\mathcal{C}$ over X is a *chain* if any two words of $\mathcal{C}$ are comparable. A chain $\mathcal{C}$ is said to be *maximal* in T , where T is a language over X , if $\mathcal{C}$ is contained in T and is not properly contained in any chain contained in T .

A simple application of Zorn's lemma proves that, given a language T over X and $u \in T$, there exists a chain $\mathcal{C}$ maximal in T and containing u .

Lemma 4. Let I be an ideal of X^* . Then every chain $\mathcal{C}$ maximal in $\bar{I}$ has the property that, for any $x \in \mathcal{C}$ and any natural number $n \leq |x|$, there exists $y \in \mathcal{C}$ with $|y| = n$.

Proof. Let $m = |x|$. If $m = 1$, the lemma is valid. Assume the lemma is true for a word of length $m - 1$ belonging to $\mathcal{C}$ and let $|x| = m > 1$. Let x_f and x_l be the words obtained by deleting the first and last letters of x , respectively. Clearly $(x) = \{x\} \cup (x_f) \cup (x_l)$, whence, since $(x) \subseteq \bar{I}$ by Lemma 2 and $\mathcal{C}$ is maximal, it follows that either x_f or x_l is in $\mathcal{C}$. But $|x_f| = |x_l| = m - 1$ and thus by the induction hypothesis, given a natural number $n \leq m - 1$, there is a word in $\mathcal{C}$ of length n . The result now follows by induction. $\square$

The following is the main result.

Theorem 5. Let I be an ideal of X^* . Then ρ_I is principal iff $\bar{I}$ contains at most one word which is maximal in $\bar{I}$.

Proof. Suppose first that $\bar{I}$ contains no maximal words. If $\bar{I} = \emptyset$, then $I = X^*$ and ρ_I is clearly principal. Thus assume $\bar{I} \neq \emptyset$ and let $\bar{I} = \{x_1, x_2, \dots\}$. Since $\bar{I}$ has no maximal words, each x_i belongs to an infinite chain C_i maximal in $\bar{I}$. By Lemma 4, given any natural number n , there exists a word $w \in C_i$ with $|w| = n$. Let $p_1 < p_2 < p_3 \dots$ be the sequence of primes. For each C_i , choose a subchain $D_i = \{z_1^{(i)} \leq z_2^{(i)} \leq \dots\}$ such that $|z_j^{(i)}| = 2^{p_j}$ and let $L = \bigcup_i D_i$. Let $u, v \in L$ and let $|u| = 2^{p_s}$, $|v| = 2^{p_t}$. If $|u| = |v|$, then $p_s^i = p_t^j$ whence $p_s = p_t$ and $i = j$. Therefore $s = t$ and $i = j$ whence $u = v$. Thus no two words of L can have the same length. Let $u, v \in L$, $u \neq v$. Then $|u| = 2^s$, $|v| = 2^t$ with $s > t$, say. Now $u \in D_i$ for some i , whence there exist $x, y \in X^*$ such that $xy \neq 1$ and $xuy \in D_i \subseteq L$. Let $|xuy| = 2^l$. Then $|x| + |y| = 2^l - 2^s$, $l > s > t$. But $|xvy| = 2^l - 2^s + 2^t = 2^t(2^{l-t} - 2^{s-t} + 1)$ and hence, since $2^{l-t} - 2^{s-t} + 1$ is odd, it follows that $xvy \notin L$. Therefore $u \neq v(P_L)$ and hence every P_L -class contained in L is a singleton class. Let $w \in \bar{I}$. Then $w \in C_i$ for some i and thus there exist $p, q \in X^*$ such that $pwq \in L$. But, denoting the P_L -class of a word x by $[x]_L$, we have $p[w]_L q \subseteq [pwq]_L = \{pwq\}$ whence $|[w]_L| = 1$. Moreover, ρ_I saturates L and so I

is contained in a P_L -class since P_L is the coarsest congruence saturating L . Since the P_L -classes in $\bar{I}$ are singletons, it follows that I is a P_L -class and hence $P_L = \rho_I$.

Assume now that $\bar{I}$ contains exactly one maximal word, say w . If $\bar{I} = (w]$, then clearly $\rho_I = P_{\{w\}}$. Thus assume that $(w] \neq \bar{I}$. Each word x_i of $\bar{I} \setminus (w] = \{x_1, x_2, \dots\}$ belongs to an infinite chain C_i maximal in $\bar{I}$. Choose the subchains D_i of C_i as above and let $L = (\bigcup_i D_i) \cup \{w\}$. Let $u, v \in L \setminus \{w\}$, $u \neq v$. Then $|u| = 2^s$, $|v| = 2^t$, $s > t$ and there exist $x, y \in X^*$ such that $|x| + |y| > |w|$ and $xuy \in L \setminus \{w\}$. Arguing as before and since $|x| + |y| > |w|$, it follows that $xvy \notin L$. Therefore $u \not\equiv v(P_L)$. Let $u \in L \setminus \{w\}$. Then $u \not\equiv w(P_L)$ since the only context of w is $(1, 1)$. Thus the P_L -classes contained in L are singleton classes. As above, since $(X^*uX^*) \cap L \neq \emptyset$ for all $u \in \bar{I}$, it follows that $\rho_I = P_L$.

Conversely, suppose $\bar{I}$ contains more than one maximal word, say u, v with $u \neq v$. Suppose there exists L with $\rho_I = P_L$. Since P_L saturates L , it follows that either $I \cap L = \emptyset$ or $I \subseteq L$. Assume, therefore, without loss of generality, that $L \subseteq \bar{I}$. Since u and v are maximal in $\bar{I}$, $xuy \in \bar{I}$ or $xvy \in \bar{I}$ implies $xy = 1$. If $u \notin L$ (resp. $v \notin L$), then $u \equiv w(P_L)$ (resp. $v \equiv w(P_L)$) for all $w \in I$, a contradiction. Hence $u, v \in L$ and $u \equiv v(P_L)$. This contradiction shows that ρ_I cannot be principal. $\square$

We now obtain a necessary and sufficient condition for a Rees congruence not to be principal in terms of the minimal generating set of the ideal.

Definition 6 [1]. Let I be an ideal of X^* . Then the set of minimal words of I is called the *infix root* of I and is denoted by $r(I)$.

Clearly $I = [r(I)) = X^*r(I)X^*$. It is also obvious that $r(I)$ is the minimal generating set of I .

Definition 7. The word u is a *prefix* (resp. *suffix*) of w if $w = ux$ (resp. $w = xu$) for some $x \in X^*$.

Notation. For a given language L , let $\mathcal{P}(L) = \{u | u \neq 1 \text{ and } u \text{ is a prefix of some word in } L\}$, $\mathcal{S}(L) = \{u | u \neq 1 \text{ and } u \text{ is a suffix of some word in } L\}$.

Corollary 8. Let I be an ideal of X^* . Then ρ_I is not principal iff there exist distinct words u and v in $\bar{I}$ such that

$$\begin{aligned} |\mathcal{P}(Xu) \cap r(I)| &= |\mathcal{S}(uX) \cap r(I)| \\ &= |\mathcal{P}(Xv) \cap r(I)| \\ &= |\mathcal{S}(vX) \cap r(I)| \\ &= |X|. \end{aligned}$$

Proof. If ρ_I is not principal, by Theorem 5 $\bar{I}$ contains two distinct words u and v which are maximal in $\bar{I}$. Clearly $(uX) \cup (Xu) \cup (vX) \cup (Xv) \subseteq I$. Let $a \in X$. Then there exists $w \in r(I)$ such that $w \leq au$, whence $w = au'$ where $u'u'' = u$.

Similar statements apply to ua , av , and va and the necessity of the result follows.

Conversely, if the condition holds, u and v are maximal in $\bar{I}$ and thus by Theorem 4, ρ_I is not principal. $\square$

Recall that an ideal I of a semigroup S is prime, if for all $a, b \in S$, $aSb \subseteq I$ implies $a \in I$ or $b \in I$. We are now able to prove the following.

Theorem 9. *I is either a prime ideal or $|r(I)| < |X|$, then ρ_I is principal. Hence, if I is principal, ρ_I is principal.*

Proof. If I is prime, then given $u, v \notin I$, there exists $x \in X^*$ such that $uxv \notin I$. Hence $\bar{I}$ contains no maximal word whence, by Theorem 5, ρ_I is principal. If $|r(I)| < |X|$, $|\mathcal{P}(Xu) \cap r(I)| < |X|$ for all $u \in X^*$. Therefore, by Corollary 8, ρ_I is principal. $\square$

Remark. It can easily be proved that if $|X| \geq 3$ all principal ideals are prime. However, if $|X| = 2$, X^*uX is prime iff $u \notin ab^+ \cup a^+b \cup ba^+ \cup b^+a$ where $X = \{a, b\}$.

Definition 10. An infix code L is said to be maximal if $L \cup \{w\}$ is not an infix code for any $w \in X^* \setminus L$.

Theorem 11. *Let X be an alphabet with more than one letter and let I be an ideal of X^* such that $r(I)$ is a finite maximal infix code. Then ρ_I is not principal provided $r(I) \neq X$.*

Proof. Since $r(I)$ is finite and maximal, $\bar{I}$ is also finite and hence contains words which are maximal in $\bar{I}$. Thus, by Theorem 5 $\bar{I}$ contains exactly one maximal word w if ρ_I is principal. Clearly $\bar{I} = (w]$ and $\bar{I} \neq \{1\}$ since $r(I) \neq X$. By the maximality of $r(I)$, there exists $t \in r(I)$ such that $w \leq t$. Letting t_f and t_l denote the words obtained by deleting the first and last letters of t , respectively, we have $t_f \leq w \leq t$ and $t_l \leq w \leq t$. Hence $t_f = w = t_l$ whence $w = a^s$, $s > 0$ for some $a \in X$ and $t = a^{s+1}$. It follows that each word of $r(I)$ is a power of the letter a . Hence $r(I) = \{a^{s+1}\}$, contradicting the maximality of the infix code $r(I)$. Hence ρ_I is not principal. $\square$

The above theorem is not valid if $|r(I)| = \infty$ as the following example shows.

Example 12. Let $X = \{a, b\}$ and let $w = a_1^{n_1} a_2^{n_2} \cdots a_s^{n_s} \neq 1$ where $n_i > 0$ and $a_i \neq a_{i+1}$ for all i . Then the skeleton of w , denoted $\text{sk}(w)$, is the word $a_1 a_2 \cdots a_s$. Let

$$L = \{ab^i a^j b | i, j > 0\} \cup \{ba^i b^j a | i, j > 0\} \cup \{a^2 b^j a^2 | j > 0\} \cup \{b^2 a^j b^2 | j > 0\}.$$

L is clearly an infix code. That it is maximal is a consequence of the following: first, if $|\text{sk}(w)| \leq 2$, clearly $w \in (L]$; if $|\text{sk}(w)| = 3$, $w = a^i b^j a^k$ or $w = b^i a^j b^k$. In either case, if $i, k \geq 2$, $w \in (L]$, while if one of i and k is 1, $w \in (L]$. If $|\text{sk}(w)| \geq 4$, $w \in (L]$ thus proving that L is indeed maximal infix.

It is easy to check that

$$\overline{X^*LX^*} = \{w \mid |\text{sk}(w)| \leq 2\} \cup \{ab^ja^k, ba^jb^k \mid j, k > 0\} \\ \cup \{a^ib^ja, b^ia^jb \mid i, j > 0\} \cup \{1\}$$

and that no word of $\overline{X^*LX^*}$ is maximal in $\overline{X^*LX^*}$. By Theorem 5, it follows that the Rees congruence modulo the ideal X^*LX^* is principal.

Theorem 13. *Let I be an ideal of X^* such that ρ_I is principal and let $L \subseteq \bar{I}$ be a language with $\rho_I = P_L$. Let F be a finite subset of L not containing the maximal word of $\bar{I}$ (if such exists). Then $P_{L \setminus F} = \rho_I$.*

Proof. Let $T = L \setminus F$ and let $u, v \in \bar{I}$. Assume first that $\bar{I}$ contains no maximal word. Then u belongs to an infinite chain in $\bar{I}$. Hence there exist $p, q \in X^*$ such that $|puq|$ and $|pvq|$ are greater than $\max\{|w| \mid w \in F\}$ and $puq \in \bar{I}$. Let $u \equiv v(P_T)$. Then $puq \equiv pvq(P_T)$. Let $xpuqy \in L$. Since $|xpuqy| > \max\{|w| \mid w \in F\}$, it follows that $xpuqy \in T$, whence $xpvqy \in T \subseteq L$. Similarly, $xpvqy \in L$ implies $xpuqy \in L$, thus proving that $puq \equiv pvq(P_L)$. Hence $puq = pvq$ and $u = v$. Therefore $\rho_I = P_T$. Suppose now that $\bar{I}$ contains a maximal word y and let $u, v \in \bar{I}$. If one of u and v belongs to an infinite chain of $\bar{I}$, the same argument as above shows that $u \equiv v(P_T)$ implies $u = v$. Suppose therefore that neither word belongs to an infinite chain of $\bar{I}$. Then $u \leq y$, $v \leq y$ and thus there exist $p, q \in X^*$ such that $puq = y$. Since $y \notin F$, it follows that $puq \in T$. Hence, if $u \equiv v(P_T)$, then $pvq \in T$. If $pvq = y$, then $u = v$. Suppose if possible that $pvq \neq y$. If $pvq \leq y$, there exist $s, t \in X^*$, $|st| > 1$ such that $spvqt = y \in T$. Hence $spuqt = syt \in T \subseteq \bar{I}$, a contradiction since y is maximal in $\bar{I}$. Thus y and pvq are incomparable. Therefore pvq belongs to an infinite chain of $\bar{I}$. An argument similar to the one given above proves $puq = pvq = y$, a contradiction. Thus P_T is equality on $\bar{I}$. Clearly I is a P_T -class, whence $P_T = \rho_I$. $\square$

We conclude by giving an example to show that the above theorem is not valid if $\bar{I}$ contains a maximal word y and $y \in F$.

Example 14. Let $X = \{a, b\}$, $\bar{I} = a^* \cup (a^2b^2)$. Then $\rho_I = \rho_L$ where $L = \{a^{2^n} \mid n \geq 1\} \cup (a^2b^2)$. However $P_{L \setminus \{a^2b^2\}} \neq \rho_I$. In fact $ab^2 \equiv a^2b(P_{L \setminus \{a^2b^2\}})$ and $ab^2, a^2b \in \bar{I}$.

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