ON OSCILLATIONS OF UNBOUNDED SOLUTIONS

I. GYÖRI, G. LADAS AND L. PAKULA

(Communicated by Kenneth L. Meyer)

ABSTRACT. Consider the differential equation with deviating arguments

$$(1) \quad \dot{y}(t) + p[y(t - \sigma_1) - y(t - \sigma_2)] = 0$$

where p, σ_1 and σ_2 are real numbers. We prove that every unbounded solution of (1) oscillates if and only if the characteristic equation

$$(2) \quad \lambda + p(e^{-\lambda\sigma_1} - e^{-\lambda\sigma_2}) = 0$$

has no positive roots and 0 is a simple root of (2).

1. INTRODUCTION

Consider the differential equation with deviating arguments

$$(1) \quad \dot{y}(t) + p[y(t - \sigma_1) - y(t - \sigma_2)] = 0$$

where p, σ_1 and σ_2 are real numbers. The characteristic equation of (1) is

$$(2) \quad F(\lambda) \equiv \lambda + p(e^{-\lambda\sigma_1} - e^{-\lambda\sigma_2}) = 0.$$

Our aim in this paper is to prove the following result.

Theorem 1. *The following statements are equivalent:*

- (a) *Every unbounded solution of equation (1) oscillates.*
- (b) *Equation (2) has no positive roots and 0 is a simple root of equation (2).*

As is customary, a solution of (1) is called *oscillatory* if it has arbitrarily large zeros. Otherwise it is called *nonoscillatory*.

We note that any nonzero constant is a (bounded, nonoscillatory) solution of (1) no matter what the values of p, σ_1 and σ_2 , and hence we do not expect an analogous equivalence statement for bounded solutions.

When σ_1 and σ_2 are both delays, or both advanced arguments, condition (b) is equivalent to a simple condition on the parameters p, σ_1 and σ_2 . (See (4), (7) below.) When σ_1 and σ_2 are both delays an elementary proof is available

Received by the editors July 31, 1988 and, in revised form, December 15, 1988.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 34K15.

The first author is on leave from the Computing Centre of the Szeged University of Medicine, 6720 Szeged, Hungary.

and we present this first. If σ_1 and σ_2 are mixed arguments then there is no such simple condition on p, σ_1 and σ_2 equivalent to (b). When at least one of the arguments σ_1 or σ_2 is advanced, we first show that a positive solution of (1) has a Laplace transform and then show, by complex integration techniques, that such a solution cannot be unbounded.

2. PROOF OF THEOREM 1

We first show (a) $\Rightarrow$ (b). Equation (2) cannot have a positive root λ for otherwise, $e^{\lambda t}$ would be an unbounded and nonoscillatory solution of (1). Also, 0 must be simple root of (2). Otherwise, $F'(0) = 0$ and so $1 + p(-\sigma_1 + \sigma_2) = 0$. In this case, (1) has the unbounded and nonoscillatory solution $y(t) = t$, which is a contradiction.

Next we show that (b) $\Rightarrow$ (a). If $p = 0$ or if $\sigma_1 = \sigma_2$, equation (1) has no unbounded solutions and so the proof is complete. Therefore we will assume that $p(\sigma_1 - \sigma_2) \neq 0$.

Without loss of generality we will also assume that $\sigma_1 > \sigma_2$.

Suppose, for the sake of contradiction, that (1) has an eventually positive and unbounded solution $y(t)$. As (1) is autonomous, we may (and will) assume that

$$y(t) > 0 \quad \text{for } t \geq \min\{0, -\sigma_1\}.$$

Set

$$z(t) = y(t) - p \int_{t-\sigma_1}^{t-\sigma_2} y(s) ds \quad \text{for } t \geq 0.$$

Then $\dot{z}(t) = 0$ and so

$$(3) \quad y(t) = C + p \int_{t-\sigma_1}^{t-\sigma_2} y(s) ds \quad \text{for } t \geq 0,$$

where

$$C = y(0) - p \int_{-\sigma_1}^{-\sigma_2} y(s) ds$$

is some constant. As $y(t)$ is positive and unbounded, it follows from (3) that $p > 0$.

We have the following two cases to consider.

Case 1. $\sigma_2 \geq 0$ (also by hypothesis $\sigma_1 > \sigma_2$).

As 0 is a simple root of equation (2) and because $F(\infty) = \infty$, it follows that $F'(0) > 0$, that is

$$(4) \quad p(\sigma_1 - \sigma_2) < 1.$$

In view of the assumption that $y(t)$ is unbounded, it follows that there exists a sequence of points $\{t_n\}$ such that

$$(5) \quad \lim_{n \rightarrow \infty} t_n = \infty, \quad \lim_{n \rightarrow \infty} y(t_n) = \infty \quad \text{and} \quad y(t_n) = \max_{s \leq t_n} y(s)$$

for $n = 1, 2, \dots$. Then (3) yields,

$$y(t_n) = C + p \int_{t_n - \sigma_1}^{t_n - \sigma_2} y(s) ds \leq C + p(\sigma_1 - \sigma_2)y(t_n)$$

or

$$[1 - p(\sigma_1 - \sigma_2)]y(t_n) \leq C,$$

which, in view of (4) and (5) leads to a contradiction as $n \rightarrow \infty$.

Case 2. $\sigma_2 < 0$ (also by hypothesis $\sigma_1 > \sigma_2$).

The proof in this case will proceed by a series of lemmas. For the remainder of this proof one should recall that $y(t)$ has been assumed positive and unbounded, and we are seeking a contradiction.

Lemma 1. *There exists a positive number c such that $y(t) = O(e^{ct})$.*

Proof. We will first show that, for some c ,

$$(6) \quad \int_0^\infty e^{-ct} y(t) dt < \infty.$$

As 0 is a simple root of (2) and because $F(\infty) = -\infty$, it follows that $F'(0) < 0$, that is,

$$(7) \quad p(\sigma_1 - \sigma_2) > 1.$$

Set $y(t) = z(t) + k$ where $k = C/[1 - p(\sigma_1 - \sigma_2)]$ and C is the constant in (3). Then (3) yields

$$(8) \quad z(t) = p \int_{-\sigma_1}^{-\sigma_2} z(t+s) ds$$

and

$$(9) \quad z(t) \geq -K \quad \text{where } K = |k|.$$

Now fix an $\varepsilon > 0$ with $-\sigma_1 < \varepsilon < -\sigma_2$ and choose $c > 0$ so that

$$(10) \quad \gamma \equiv p \int_\varepsilon^{-\sigma_2} e^{cs} ds > 1.$$

We will show that for this c , the function $y(t)$ satisfies (6). To this end set

$$(11) \quad w(t) = \int_0^t e^{-cu} z(u) du$$

and observe that

$$\begin{aligned} w(t) &= \int_0^t p e^{-cu} \int_{-\sigma_1}^{-\sigma_2} z(u+s) ds du \\ &= p \int_{-\sigma_1}^{-\sigma_2} \int_0^t e^{-cu} z(u+s) du ds \\ &= p \int_{-\sigma_1}^{-\sigma_2} e^{cs} \int_s^{t+s} e^{-cv} z(v) dv ds. \end{aligned}$$

That is,

$$(12) \quad w(t) = p \int_{-\sigma_1}^{-\sigma_2} e^{cs} w(t+s) ds - C_1$$

where

$$C_1 = p \int_{-\sigma_1}^{-\sigma_2} e^{cs} w(s) ds.$$

Now observe that by (9) and (11)

$$(13) \quad w(t) \geq -K/c \quad \text{for } t \geq 0$$

and so (12) yields,

$$(14) \quad \begin{aligned} w(t) &= p \int_{-\sigma_1}^{-\sigma_2} e^{cs} [w(t+s) + K/c] ds - \frac{Kp}{c} \int_{-\sigma_1}^{-\sigma_2} e^{cs} ds - C_1 \\ &\geq p \int_{\varepsilon}^{-\sigma_2} e^{cs} w(t+s) ds - C_2, \end{aligned}$$

where $C_2 = Kp(e^{-\sigma_2 c} - e^{-\sigma_1 c})/c^2 - C_1$.

By the mean value theorem for integrals and with γ as chosen in (10), it follows from (14) that given any $t_0 > 0$ there exists a $t_1 \in (t_0 + \varepsilon, t_0 - \sigma_2)$ such that $w(t_0) + C_2 \geq \gamma w(t_1)$. By induction there is a sequence $\{t_n\}$ such that, for $n = 0, 1, \dots$,

$$(15) \quad \begin{aligned} t_n + \varepsilon &< t_{n+1} < t_n - \sigma_2; \\ w(t_{n+1}) &\leq w(t_n)/\gamma + C_2/\gamma. \end{aligned}$$

Consider the difference equation

$$x_{n+1} = x_n/\gamma + C_2/\gamma$$

for $n = 0, 1, \dots$, with $x_0 = w(t_0)$. It follows by induction that

$$(16) \quad w(t_n) \leq x_n.$$

Clearly

$$\lim_{n \rightarrow \infty} x_n = C_2/(\gamma - 1)$$

and so from (16) and (13) we conclude that the sequence $\{w(t_n)\}$ is bounded above by a constant, say M . Then for $n = 0, 1, \dots$ we have

$$\begin{aligned} \int_0^{t_n} e^{-cs} y(s) ds &= \int_0^{t_n} e^{-cs} [z(s) + k] ds \\ &= w(t_n) + k \int_0^{t_n} e^{-cs} ds \\ &\leq M + K/c. \end{aligned}$$

It follows that the Laplace transform

$$(17) \quad Y(s) = \int_0^\infty e^{-st} y(t) dt$$

of the positive function $y(t)$ exists for $\operatorname{Re} s > c$. Now multiply (3) by e^{-ct} and note that for s in the interval of integration we have $e^{-ct} \leq e^{-cs} e^{-c\sigma_2}$. It now follows from the finiteness of integral (17) that $e^{-ct}(y(t) - C) \rightarrow 0$ which establishes the lemma.

Define $\phi(s)$ for complex s by

$$(18) \quad \phi(s) = y(0) - pe^{-s\sigma_1} \int_{-\sigma_1}^0 e^{-st} y(t) dt + pe^{-s\sigma_2} \int_{-\sigma_2}^0 e^{-st} y(t) dt.$$

Taking the Laplace transform of both sides of (1) we get, for $\operatorname{Re} s > c$, that

$$F(s)Y(s) = \phi(s).$$

Fix a real number $a > c$. By the inversion theorem for Laplace transforms (as in [4], p. 108) we have

$$y(t) = \lim_{r \rightarrow \infty} \frac{1}{2\pi i} \int_{a-ir}^{a+ir} e^{st} \phi(s)/F(s) ds.$$

For notational convenience we will abbreviate such integrals by

$$y(t) = \int_{(a)} e^{st} \phi(s)/F(s) ds.$$

We will prove that

$$(19) \quad y(t) = L(t) + R(t)$$

where $R(t)$ is a constant plus a finite number of sinusoids, and $L(t) \rightarrow 0$ as $t \rightarrow \infty$. We use techniques similar to those of §§4.2-4 and 12.15 of [1].

Lemma 2. *For any real number b , the characteristic function $F(s)$ has only finitely many roots in the vertical strip $b \leq \operatorname{Re} s < a$.*

Proof. From the definition of $F(s)$ it is easy to see that for large enough $r > 0$ we have $|F(x \pm ir)| > r/2$ for $b \leq x < a$. In particular, there is an $M > 0$ so that if $b \leq \operatorname{Re} s < a$ and $F(s) = 0$, then $|\operatorname{Im} s| < M$, i.e. all zeros in the vertical strip actually lie in a finite rectangle. By analyticity of F there are then only finitely many zeros in the strip.

Fix $a > c$ where c is as in Lemma 1. It follows from Lemma 2 that we can choose a number $b < 0$ so that in the strip $b \leq \operatorname{Re} s < a$, $\phi(s)/F(s)$ has only finitely many poles. In particular we can assume that $F(s) \neq 0$ for $\operatorname{Re} s = b$. Let $R(t)$ be the sum of the residues of $e^{st} \phi(s)/F(s)$ arising from the poles in $b \leq \operatorname{Re} s < a$ and let

$$(20) \quad L(t) = \int_{(b)} -e^{st} \phi(s)/F(s) ds.$$

We fix a value of t for now and verify (19). To this end, consider the rectangular contour γ with vertices $a \pm ir$, $b \pm ir$. Denote the vertical segments by γ_1 and γ_3 , and the horizontal segments by γ_2 and γ_4 . Then we claim

$$\lim_{r \rightarrow \infty} \int_{\gamma_k} e^{st} \phi(s)/F(s) ds = 0$$

for $k = 2, 4$. Recall that t is fixed here. It is easy to check that there is an $m > 0$ such that $|e^{st}\phi(s)| < m$ for $s = x \pm ir$ with $b \leq x \leq a$ and any r . By the argument of Lemma 2, we see that $|1/F(x \pm ir)| < 2/r$ for all $b \leq x \leq a$ and all large enough r . Thus for $k = 2, 4$ we have

$$\int_{\gamma_k} e^{st}\phi(s)/F(s) ds = O(1/r)$$

as $r \rightarrow \infty$ which establishes the claim.

Thus (19) follows from the residue theorem and the inversion theorem. Next we show that $L(t) \rightarrow 0$ as $t \rightarrow \infty$. Let us write $\phi(s) = \phi_0(s) + \phi_1(s) + \phi_2(s)$ corresponding to the three terms on the right in (18). We show

$$(21) \quad \int_{(b)} e^{st}\phi_k(s)/F(s) ds \rightarrow 0$$

as $t \rightarrow \infty$, for $k = 0, 1$ and 2 .

First consider $k = 1$. Then $\phi_0(s)$ is the constant $y(0)$ which we take to be 1 for convenience. Define $f(w) = F(b + iw)$ for real w and let

$$(22) \quad g(t) = \int_{(b)} e^{st}/F(s) ds = \lim_{r \rightarrow \infty} e^{bt} \int_{-r}^r e^{iwt}/f(w) dw.$$

Also, define

$$T(w) = \begin{cases} 1/iw, & \text{if } |w| \geq 1; \\ 0, & \text{otherwise.} \end{cases}$$

From our assumption on b , $f(w) \neq 0$ for all w . We see from the explicit form of $f(w)$ that $|1/f(w) - T(w)| = O(1/w^2)$ as $w \rightarrow \infty$ and we thus conclude that $1/f(w) - T(w)$ is absolutely integrable so that

$$(23) \quad \lim_{r \rightarrow \infty} \int_{-r}^r e^{iwt}[1/f(w) - T(w)] dw = \int_{-\infty}^{\infty} e^{iwt}[1/f(w) - T(w)] dw.$$

This last integral is bounded in t . On the other hand, explicit contour integration shows that as $t \rightarrow \infty$,

$$(24) \quad \lim_{r \rightarrow \infty} \int_{-r}^r e^{iwt} T(w) dw \sim \frac{1}{t}.$$

Now combining (23) and (24), and using $b < 0$, we see from (22) that $g(t) \rightarrow 0$ as $t \rightarrow \infty$.

Next, consider ϕ_k with $k = 1$ or 2 . We take $k = 1$ to be specific. Again, let $f(w) = F(b + iw)$ for real w , but let

$$(25) \quad g(t) = \int_{(b)} e^{st}\phi_1(s)/F(s) ds = \lim_{r \rightarrow \infty} e^{bt} \int_{-r}^r e^{iwt}\Phi_1(w)/f(w) dw$$

where $\Phi_1(w) = \phi_1(b + iw)$. A change of variable in the definition of $\phi_1(s)$ gives

$$(26) \quad \Phi_1(w) = \int_{-2\sigma_2}^{-\sigma_1} e^{i w \tau} e^{b \tau} y(\tau + \sigma_1) d\tau.$$

We see from this that Φ_1 is the Fourier transform of a bounded function of τ with compact support, that function thus being in both $L^1(d\tau)$ and $L^2(d\tau)$. But the ordinary Fourier transform of such a function is in $L^2(dw)$ ([3], p. 187). Since $f(w) \neq 0$ and $|1/f(w)| = O(1/|w|)$ as $w \rightarrow \infty$, we conclude that $1/f(w)$ is in $L^2(dw)$. As the product of two functions in $L^2(dw)$ is in $L^1(dw)$ it follows that $\Phi_1(w)/f(w) \in L^1$. Hence

$$(27) \quad g(t) = e^{bt} \int_{-\infty}^{\infty} e^{iwt} \Phi_1(w)/f(w) dw$$

and as in the previous case we can show $g(t) \rightarrow 0$ as $t \rightarrow \infty$.

We have thus shown that $L(t) \rightarrow 0$ in (19). Let the distinct zeros of $F(s)$ in the strip $b \leq \operatorname{Re} s < a$ be denoted by $s_0, s_1, \dots, s_{2n}$, with $s_0 = 0$. Note that the other zeros are in conjugate pairs. Then $R(t)$ has the form

$$\sum_k P_k(t) e^{s_k t}$$

where $P_k(t)$ is a polynomial (with complex coefficients) of degree less than the multiplicity of the zero s_k [1, p. 108]. Since 0 is a simple root of (2) by hypothesis, $\deg P_0 = 0$, so the term corresponding to $k = 0$ is a constant.

If $n = 0$ then $R(t)$ is constant and $y(t)$ is bounded, a contradiction. Otherwise, the real part of $R(t)$ can be written as

$$(28) \quad \text{const} + \sum_{k=1}^n e^{r_k t} (p_k(t) \cos \theta_k t + q_k(t) \sin \theta_k t)$$

where $p_k(t)$ and $q_k(t)$ are polynomials with real coefficients and $r_k \geq 0$. We need the following lemma. A simple proof is included for the reader's convenience.

Lemma 3. *For any sum*

$$f(t) = \sum_{k=1}^n a_k \cos \theta_k t + b_k \sin \theta_k t$$

which is not identically zero and for which all $\theta_k \neq 0$, there exists $\varepsilon > 0$ such that $\limsup_{t \rightarrow \infty} f(t) > \varepsilon$ while $\liminf_{t \rightarrow \infty} f(t) < -\varepsilon$.

Proof. Since $f(t)$ is an almost-periodic function, the result follows easily from the (Bohr) definition of almost periodicity (see e.g. [2]) if we can show that $f(t)$ takes on both positive and negative values. Observe that any $f(t)$ as in the lemma has mean value

$$M(f) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(t) dt$$

zero. Suppose, say, $f(t) \geq 0$ for all t . Then $f(t)$ would have an indefinite integral $F(t)$ of the same form as f , with $M(F) = 0$, and with $F(t)$ nondecreasing and bounded. Then $\lim_{t \rightarrow \infty} F(t) = L$ for some L , so $M(F) = L = 0$. But,

assuming as we may, all θ_k distinct and positive, $M(F^2) = \sum(a_k^2 + b_k^2)/2\theta_k^2 > 0$ by our assumptions on f , so $0 < M(F^2) = L^2$; a contradiction.

In (28), let r_M be the largest of the r_k 's and let m be the highest degree among the p_k 's and q_k 's with $r_k = r_M$. Letting α_k, β_k be the coefficients of t^m in p_k, q_k , set

$$R_1(t) = e^{r_M t} t^m \sum \alpha_k \cos \theta_k t + \beta_k \sin \theta_k$$

where the sum is over those k for which $r_k = r_M$. If $r_M > 0$ or if $r_M = 0$ but $m > 0$ we could find, using Lemma 3, a sequence $t_n \rightarrow \infty$ for which $\liminf_{n \rightarrow \infty} R_1(t_n) = -\infty$ while $R(t_n) \sim R_1(t_n)$ as $n \rightarrow \infty$. In view of (19), this contradicts the positivity of $y(t)$. Thus $r_M = 0$, $m = 0$, and $R(t)$ is bounded; hence $y(t)$ is bounded. This contradiction completes the proof of the theorem.

REFERENCES

1. R. Bellman and K. Cooke, *Differential-difference equations*, Academic Press, New York, 1963.
2. C. Corduneanu, *Almost-periodic functions*, Interscience Publishers, New York, 1968.
3. W. Rudin, *Real and complex analysis*, McGraw-Hill, New York, 1966.
4. D. V. Widder, *An introduction to transform theory*, Academic Press, New York, 1971.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF RHODE ISLAND, KINGSTON, RHODE ISLAND 02881