AN IDENTITY IN THE FREE LIE ALGEBRA

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ABSTRACT. An identity in the free Lie algebra is shown to imply the theorems of Friedrichs and Specht-Wever.

Let $\mathcal{L}(V)$ denote the free Lie algebra on the vector space V over the field k . By 'abstract nonsense', we may identify the universal enveloping algebra $U(\mathcal{L}(V))$ with the tensor algebra $\mathcal{T}(V)$ on V . The algebra $\mathcal{T}(V)$ possesses a natural grading, $\mathcal{T}(V) = \bigoplus_0^\infty \mathcal{T}^n(V)$. We have $\mathcal{T}^0(V) = k$ and $\mathcal{T}^1(V) = V$. (We consider V as embedded in $\mathcal{L}(V)$ and $\mathcal{L}(V)$ as embedded in $\mathcal{T}(V)$.) Let Δ denote the diagonal map of $\mathcal{T}(V)$ and μ the multiplication. Let $\gamma : \mathcal{T}(V) \rightarrow \mathcal{L}(V)$ be the map defined by the formula $\gamma(v_1 \dots v_n) = [v_1[v_2 \dots [v_{n-1}, v_n] \dots]]$ for $v_1, \dots, v_n \in V$. We define $\gamma|_{\tau^0(V)} = 0$ and $\gamma|_{\tau^1(V)} = \text{Id}_V$. If $x \in \mathcal{T}(V)$ and $y \in \bigoplus_1^\infty \mathcal{T}^n(V)$, then $\gamma(xy) = \text{ad}(x)(\gamma(y))$.

Proposition. If $x \in \mathcal{T}^n(V)$, then

$$\mu \circ (\gamma \otimes \text{Id}_{\mathcal{T}(V)}) \circ \Delta(x) = nx.$$

Proof. The proof is by induction on n , the cases $n = 0$ and $n = 1$ being clear. Let $x = v_1 \dots v_n$ with $v_1, \dots, v_n \in V$. We may write $\Delta(v_2 \dots v_n) = 1 \otimes v_2 \dots v_n + \sum a_i \otimes b_i$ with each $a_i \in \bigoplus_1^\infty \mathcal{T}^n(V)$. The inductive hypothesis gives $\sum \gamma(a_i) \cdot b_i = (n-1)v_2 \dots v_n$. We have

$$\begin{aligned} \mu \circ (\gamma \otimes \text{Id}_{\mathcal{T}(V)}) \circ \Delta(v_1 \dots v_n) &= \mu \circ (\gamma \otimes \text{Id}_{\mathcal{T}(V)}) \left\{ (v_1 \otimes 1 + 1 \otimes v_1) \left(1 \otimes v_2 \dots v_n \right. \right. \\ &\quad \left. \left. + \sum a_i \otimes b_i \right) \right\} \\ &= v_1 \dots v_n + \sum \{ \text{ad}(v_1)(\gamma(a_i)) \cdot b_i + \gamma(a_i) \cdot v_1 \cdot b_i \} \\ &= v_1 \dots v_n + \sum v_1 \cdot \gamma(a_i) \cdot b_i \\ &= v_1 \dots v_n + v_1 \cdot (n-1)v_2 \dots v_n \\ &= nv_1 \dots v_n. \end{aligned}$$

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From the proposition we may easily deduce the theorems of Friedrichs ([1, p. 33], [2, p. 170]) and Specht-Wever ([1, p. 34], [2, p. 169]).

Theorem. *Suppose that k has characteristic zero. If $x \in \mathcal{T}^n(V)$, then the following are equivalent:*

- (1) $x \in \mathcal{L}(V)$;
- (2) $\Delta(x) = 1 \otimes x + x \otimes 1$;
- (3) $\gamma(x) = nx$.

Proof. (1) $\Rightarrow$ (2). Since (2) is true for an element of degree 1, and if $\Delta(x) = 1 \otimes x + x \otimes 1$ and $\Delta(y) = 1 \otimes y + y \otimes 1$, then $\Delta([x, y]) = 1 \otimes [x, y] + [x, y] \otimes 1$, we get (2) for elements of $\mathcal{L}(V)$ by induction on the degree.

(2) $\Rightarrow$ (3). From (2) and the proposition we have

$$\begin{aligned} nx &= \mu \circ (\gamma \otimes 1) \circ \Delta(x) \\ &= \mu \circ (\gamma \otimes 1)(1 \otimes x + x \otimes 1) \\ &= \gamma(x). \end{aligned}$$

(3) $\Rightarrow$ (1) is clear.

REFERENCES

1. N. Bourbaki, *Groupes et algèbres de Lie*, Chapitres 2 et 3, Hermann, Paris 1972.
2. N. Jacobson, *Lie algebras*, Interscience, New York, 1962.

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