WEAKLY DENSE SUBSETS OF THE MEASURE ALGEBRA

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ABSTRACT. We introduce the notion of the weak density of a Boolean algebra and show that for homogeneous measure algebras it coincides with the density (=least size of a coinital set). From this we obtain a partial lifting of the measure algebra of $[0, 1]$ of minimal size which does not extend to a lifting. It also follows that the π -character of each point and the π -weight are the same for the Stone space of a homogeneous measure algebra

0. NOTATION

For set-theoretic and topological notation not described below, see [3, 8, 9, 10]. For measure theory see [7]. If B is a Boolean algebra and $a \in B$, a^c is the complement of a . $B_a = B|a = \{b \in B : b \leq a\}$. B is *homogeneous* if B_a is isomorphic to B for all $a \in B \setminus \{0\}$. If f is a function which assigns an ordinal to each Boolean algebra, then B is *homogeneous in f* if $f(B_a) = f(B)$ for all $a \in B \setminus \{0\}$. $\text{St}(B)$ is the Stone space of B . If $B' \subseteq B$ is a subalgebra and $a \in B$ then $\langle B' \cup \{a\} \rangle = \{(b \wedge a) \vee (c \setminus a) : c, b \in B'\}$ is the subalgebra generated by $B' \cup \{a\}$. For any infinite set K , $[K]^\omega$ is the partial order of countably infinite subsets of K , ordered by inclusion. $cf[K]^\omega = \min\{|D| : D \subseteq [K]^\omega \text{ and } (\forall B \in [K]^\omega)(\exists A \in D)(B \subseteq A)\}$.

If (X, Σ, μ) is a probability space and $A, B \in \Sigma$ then we write $A \subseteq^* B$, $A =^* B$ to mean $\mu(A \setminus B) = 0$, $\mu(A \Delta B) = 0$, respectively. If I is any nonempty set, then (X^I, μ) is the product space with product measure denoted also by μ . For $a \in I$ let $\pi_a : X^I \rightarrow X^{I \setminus \{a\}}$ be defined by $\pi_a(f) = f|_{I \setminus \{a\}}$. Then for any set $E \subseteq X^I$, $\text{supp}(E) = \{a \in I : E \neq \pi_a^{-1} \pi_a'' E\}$.

For cardinals $\kappa \geq \omega$, μ is the usual product measure on $\{0, 1\}^\kappa$, Σ_κ is the σ -algebra of measurable sets, and $\mathcal{N}_\kappa = \{E \in \Sigma_\kappa : \mu E = 0\}$. $\mathcal{A}_\kappa = \Sigma_\kappa / \mathcal{N}_\kappa$ is the measure algebra of $(\{0, 1\}^\kappa, \Sigma_\kappa, \mu)$. If $E \in \Sigma_\kappa$ then $[E] \in \mathcal{A}_\kappa$ is the equivalence class of E . If B is a subalgebra of $\mathcal{A}_\kappa$, then a *partial lifting* $\rho : B \rightarrow \Sigma_\kappa$ is a Boolean homomorphism satisfying $\rho(a) \in a$ for each $a \in B$. If $[U] \in B$, and $\rho[U] \supseteq U$ for each clopen Baire set U , then ρ is called a

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strong partial lifting. If $B = \mathcal{A}_\kappa$, ρ is called a *lifting*. For finite functions s , let $[s] = \{f \in \{0, 1\}^\kappa : s \subseteq f\}$; if $x \in E \in \Sigma_\omega$, the *density* of E at x is $d(x, E) = \lim_{n \rightarrow \infty} \mu(E \cap [x|n]) / \mu([x|n])$ when this limit exists. For almost all $x \in E$, $d(x, E) = 1$ (see e.g. [12]).

1. INTRODUCTION

Definition. Let B be a Boolean algebra. Then $D \subseteq B \setminus \{0\}$ is *dense* if $(\forall b \in B \setminus \{0\})(\exists d \in D)(d \leq b)$. D is *weakly dense* if $(\forall b \in B \setminus \{0\})(\exists d \in D)(d \leq b \text{ or } d \leq b^c)$. Let $d(B) = \min\{|D| : D \subseteq B \text{ is dense}\}$, $\text{wd}(B) = \min\{|D| : D \subseteq B \text{ is weakly dense}\}$.

When $B = \mathcal{A}_\kappa$ for some κ , we will say $D \subseteq \Sigma_\kappa$ is *dense*-(*weakly dense*) in $\mathcal{A}_\kappa$ if $\{[d] : d \in D\}$ is dense (weakly dense) in $\mathcal{A}_\kappa$.

Remarks. (1) $\text{wd}(B) \leq d(B)$.

(2) For any $a \in B \setminus \{0\}$, if $D \subseteq B_a$ is dense or weakly dense in B_a , then D is weakly dense in B . Thus if B is not homogeneous in d then $\text{wd}(B) < d(B)$.

(3) If B is σ -complete and homogeneous in d and $d(B) \geq \omega_1$ then $\text{wd}(B) \geq \omega_1$. (Proof: Note that the hypotheses imply that B is nonatomic. If $\{a_n : n \in \omega\} \subseteq B \setminus \{0\}$ is any countable set, inductively choose disjoint elements $b_n \in B$ satisfying $b_n \subseteq a_n$ and $a_i \setminus \bigvee_{j=1}^n b_j \neq 0$ ($i > n$). (To choose b_n note that $D = \{a_i \wedge a_n \setminus \bigvee_{j=1}^{n-1} b_j : i > n\}$ is not dense in B_{a_n} and choose $b_n \in B_{a_n} \setminus \{0\}$ which does not contain any member of D . This works.) Now write $b_n = c_n \vee d_n$, $c_n \neq 0 \neq d_n$, $c_n \wedge d_n = 0$. Let $d = \sup\{d_n : n \in \omega\}$. For each n , both $d \wedge a_n$ and $d^c \wedge a_n$ are not 0 and hence $\{a_n : n \in \omega\}$ is not weakly dense.)

(4) If $\mathcal{A}$ is any measure algebra, $\{x \in \mathcal{A} : \mu(x) = 1/2\}$ is weakly dense in $\mathcal{A}$.

(5) If U is a local π -base on B (i.e. $U \subseteq B \setminus \{0\}$ and for some ultrafilter $p \subseteq B \setminus \{0\}$, $(\forall b \in p)(\exists a \in U)(a \leq b)$), then U is weakly dense in B . Since $\pi\chi(x, \text{St}(B)) \geq \text{wd}(B)$ for each $x \in \text{St}(B)$, if $\text{wd}(B) = d(B)$ then $\pi\chi(x, \text{St}(B)) = \pi w(\text{St}(B))$ for all $x \in \text{St}(B)$. (For any topological space X and for any $x \in X$, $\pi w(X) = \min\{|\mathcal{A}| : \mathcal{A} \text{ is a collection of nonempty open sets such that every nonempty open set includes a member of } \mathcal{A}\}$, and $\pi\chi(x, X) = \min\{|\mathcal{A}| : \mathcal{A} \text{ is a collection of nonempty open sets such that every neighborhood of } x \text{ includes a member of } \mathcal{A}\}$. For more on these cardinal functions see [3 or 8].)

(6) If κ is given the discrete topology and we let $B(\kappa)$ be the clopen subsets of κ^ω (the Baire space of length κ), then $d(B(\kappa)) = \kappa$ and $\text{wd}(B(\kappa)) = \omega$ (a countable neighborhood base at any point is weakly dense). However in the completion of $B(\kappa)$ the weak density is κ . (We omit the proof. It is similar to that of remark (3) above.)

Problem. Is there a homogeneous complete Boolean algebra B such that $\text{wd}(B) < d(B)$?

In §2 of this paper we prove some lemmas that will be needed later on. In §3 we will show that for any $\kappa \geq \omega$, $\text{wd}(\mathcal{A}_\kappa) = d(\mathcal{A}_\kappa) = d(\mathcal{A}_\omega) \cdot cf[\kappa]^\omega$. (The second equality is obtained in [5, Theorem 16] by different methods.) In §4 we give an application in the theory of liftings.

2. PRELIMINARY LEMMAS

Lemma 1. *If (X, λ) , (Y, μ) , (Z, ν) are probability spaces and if $A \subseteq X \times Y$, $B \subseteq Y \times Z$ are measurable sets which satisfy $A \times Z \subseteq^* X \times B$, then there is a measurable set $C \subseteq Y$ such that*

$$A \times Z \subseteq^* X \times C \times Z \subseteq^* X \times B.$$

Proof. For $x \in X$ and $z \in Z$ define $A_x = \{y \in Y : (x, y) \in A\}$, $B_z = \{y \in Y : (y, z) \in B\}$. If $\chi_A(x, y)$, $\chi_B(y, z)$ are the characteristic functions of A , B respectively, then we have $\chi_A(x, y) \leq \chi_B(y, z)$ for a.a. $(x, y, z) \in X \times Y \times Z$. By Fubini's theorem this means that for a.a. (x, z) , for a.a. y , $\chi_A(x, y) \leq \chi_B(y, z)$. Thus for a.a. z ,

$$(1) \quad \text{for a.a. } x, \quad A_x \subseteq^* B_z.$$

Let C be a set of minimal measure s.t.

$$(2) \quad \text{for a.a. } x, \quad A_x \subseteq^* C.$$

$\mathcal{C} = \{C : (2) \text{ holds for } C\}$ is closed under countable intersections. So for example choose $C_n \in \mathcal{C}$ s.t. $C_0 \supseteq C_1 \supseteq \dots$ and $\mu C_n \searrow \inf_{C \in \mathcal{C}} \mu C$. Take $C = \bigcap_n C_n$.

By minimality of μC , for any z satisfying (1), $C \subseteq^* B_z$, which together with (2) implies $A \times Z \subseteq^* X \times C \times Z \subseteq^* X \times B$. $\square$

Lemma 2. *Let (X, μ) be a nonatomic probability space. Then there is a measurable subset $E \subseteq X^\omega$ s.t. for any D of positive measure satisfying either $D \subseteq^* E$ or $D \subseteq^* E^c$, $\text{supp}(D) = \omega$.*

Proof. Fix a sequence of measurable sets $E_n \subseteq X$ ($n \geq 1$) s.t. $0 < \mu E_n < 1$ and $\prod_{n \geq 1} \mu E_n > 0$. Inductively define $A_n \subseteq X^n$ ($n \geq 1$) by

$$A_1 = E_1,$$

$$A_{n+1} = (A_n \times E_{n+1}) \cup (A_n^c \times E_{n+1}^c).$$

By induction on $n \geq 1$ we have

$$(3) \text{ For all } m < n, \quad A_m \times \prod_{m < i \leq n} E_i \subseteq A_n \text{ and } A_m^c \times \prod_{m < i \leq n} E_i^c \subseteq A_n^c.$$

(4) For every $D \subseteq X^n$ s.t. $\mu D > 0$ and $\text{supp}(D) \neq n$ we have $\mu(D \cap A_n) > 0$ and $\mu(D \cap A_n^c) > 0$.

Proof. (3) is easy. For (4): when $n = 1$ only $D = X$ satisfies the assumptions. Assume (4) and suppose $D \subseteq X^{n+1}$, $\mu D > 0$, $\text{supp}(D) \neq n + 1$.

Case 1. $n \notin \text{supp}(D)$. Write $D = D' \times X$, $D' \subseteq X^n$. One of the sets

$$\begin{aligned} D \cap (A_n \times E_{n+1}) &= (D' \cap A_n) \times E_{n+1}, \\ D \cap (A_n \times E_{n+1}^c) &= (D' \cap A_n) \times E_{n+1}^c, \\ D \cap (A_n^c \times E_{n+1}) &= (D' \cap A_n^c) \times E_{n+1}, \\ D \cap (A_n^c \times E_{n+1}^c) &= (D' \cap A_n^c) \times E_{n+1}^c, \end{aligned}$$

must have positive measure. Thus the first two of these sets or the last two both have positive measure. Either way $\mu(D \cap A_{n+1}) > 0$ and $\mu(D \cap A_{n+1}^c) > 0$.

Case 2. For some $k < n$, $k \notin \text{supp}(D)$. One of $D \cap (X^n \times E_{n+1})$, $D \cap (X^n \times E_{n+1}^c)$ has positive measure, say $\mu(D \cap (X^n \times E_{n+1})) > 0$. (The argument for the other case is obtained by replacing E_{n+1} by E_{n+1}^c in what follows.) For $e \in E_{n+1}$, write $D_e = \{x \in X^n : (x, e) \in D\}$.

Let $S = \{e \in E_{n+1} : \mu D_e > 0\}$. Then $\mu S > 0$. Since $\text{supp}(D_e) \neq n$, by the induction hypothesis, for $e \in S$, $\mu(D_e \cap A_n) > 0$ and $\mu(D_e \cap A_n^c) > 0$. Thus $\mu(D \cap (A_n \times E_{n+1}))_e > 0$ for all $e \in S$ and so $\mu(D \cap A_{n+1}) > 0$, and similarly $\mu(D \cap A_{n+1}^c) > 0$.

Now let $E = \bigcup_n (A_n \times \prod_{i>n} E_i) \subseteq X^\omega$. Fix $m \geq 1$. Then for each $n > m$, by (1) we have $A_m^c \times \prod_{m<i \leq n} E_i \subseteq A_n^c$ and so $(A_m^c \times \prod_{i>m} E_i) \cap (A_n \times \prod_{i>n} E_i) = \emptyset$. Similarly $(A_m \times \prod_{i>m} E_i) \cap (A_n^c \times \prod_{i>n} E_i) = \emptyset$. Hence

$$(5) \quad A_n^c \times \prod_{i>n} E_i \subseteq E^c \quad \text{for all } n \geq 1.$$

To finish the proof, suppose $D \subseteq X^\omega$, $\mu D > 0$, and for some $k \in \omega$, $k \notin \text{supp}(D)$. We must show $\mu(D \cap E) > 0$ and $\mu(D \cap E^c) > 0$. Since $\mu(X^n \times \prod_{i>n} E_i) \rightarrow 1$ as $n \rightarrow \infty$, there is an $n > k$ s.t. $D \cap (X^n \times \prod_{i>n} E_i)$ has positive measure. Note that $k \notin \text{supp}(D_e)$, where $D_e = \{x \in X^n : (x, e) \in D\}$ for $e \in \prod_{i>n} E_i$. Let $S = \{e \in \prod_{i>n} E_i : \mu D_e > 0\}$. Then $\mu S > 0$ and for $e \in S$, by (4) we have $\mu(D_e \cap A_n) > 0$ and $\mu(D_e \cap A_n^c) > 0$. Since $D_e \cap A_n = (D \cap (A_n \times \prod_{i>n} E_i))_e$, $\mu(D \cap (A_n \times \prod_{i>n} E_i)) > 0$ and similarly $\mu(D \cap (A_n^c \times \prod_{i>n} E_i)) > 0$; by (5) and the definition of E these give $\mu(D \cap E^c) > 0$ and $\mu(D \cap E) > 0$ respectively. $\square$

Remark. The assumption that X is nonatomic is needed here. For example if $X = \{0, 1\}$, $\mu\{0\} = \mu\{1\} = 1/2$, then for any $E \subseteq X^\omega$ s.t. $\mu E > 0$, there is a $D \subseteq E$ s.t. $\mu D > 0$ and $\omega \setminus \text{supp}(D)$ is infinite.

Proof. The following proof is a simplification due to D. Fremlin of a proof which appeared in an earlier draft of this paper.

Let $E \subseteq X^\omega$, $\mu E > 0$. For each $I \in [\omega]^{<\omega}$, define $f_I : 2^\omega \rightarrow 2^\omega$ by $f_I(x)(n) = x(n)$ if $n \notin I$, and $1 - x(n)$ if $i \in I$. We write f_n for $f_{\{n\}}$. Since E can be approximated arbitrarily well by clopen sets C (in the sense of making $\mu(E \Delta C)$ small) and for each clopen set C , $f_n''(C) = C$ for sufficiently large

n , we have $\lim_{n \rightarrow \infty} \mu(E \Delta f_n''(E)) = 0$. Inductively define an increasing sequence $\langle a_n : n \in \omega \rangle$ of elements of ω so that $D_n = \bigcap \{f_I''(E) : I \subseteq \{a_0, \dots, a_{n-1}\}\}$ has measure $\geq \mu(E) \cdot \prod_{i < n} (1 - 1/2^{i+1})$. Then $D = \bigcap_{n < \omega} D_n = \bigcap \{f_I''(E) : I \subseteq \{a_i : i < \omega\}\}^{<\omega}$ is as desired. $\square$

Lemma 3. *If $N \in \mathcal{N}_\omega$, there is an $E \in \Sigma_\omega$ s.t.*

$$\{x : d(x, E) = 1\} \cup \{x : d(x, E^c) = 1\}$$

is disjoint from N .

Proof. See [6, Theorem 4, p. 175]. $\square$

3. WEAK DENSITY OF MEASURE ALGEBRAS

Theorem 1. *If $\kappa \geq \omega$ then $\text{wd}(\mathcal{A}_\kappa) = d(\mathcal{A}_\kappa) = d(\mathcal{A}_\omega) \cdot \text{cf}[\kappa]^\omega$.*

Proof. Case 1. $\kappa = \omega$. We must show $\text{wd}(\mathcal{A}_\omega) = d(\mathcal{A}_\omega)$. This is similar to the proof in [2] that $d(\mathcal{A}_\omega) = \text{cofinality of } \mathcal{N}_\omega \text{ (ordered by inclusion)}$. Let $D \subseteq \Sigma_\omega$ of size $\text{wd}(\mathcal{A}_\omega)$ be weakly dense in $\mathcal{A}_\omega$. It will suffice to find a cofinal set $\mathcal{B} \subseteq \mathcal{N}_\omega$ of size $|D|$. Without loss of generality assume that each $H \in D$ has density 1 at all its points. Let Q be a countable dense subset of $\{0, 1\}^\omega$. Take $\mathcal{B} = \{(\bigcup_{q \in Q} H + q)^c : H \in D\}$.

Let $N \in \mathcal{N}_\omega$. Find $H \in D$ s.t. $N_1 = \bigcup_{q \in Q} N + q$ is disjoint from H . (If E is as in Lemma 3 for N_1 , then there is an $H \in D$ s.t. $H \subseteq^* E$ or $H \subseteq^* E^c$. Since H has density 1 at all its points $H \cap N_1 = \emptyset$.) For each $q \in Q$, we have $H \cap (N + q) = \emptyset$, or equivalently $N \cap (H + q) = \emptyset$. Thus $N \subseteq (\bigcup_{q \in Q} H + q)^c \in \mathcal{B}$.

Case 2. $\kappa > \omega$. For each $A \in [\kappa]^\omega$ fix an enumeration $A = \{A(n) : n \in \omega\}$ of A and define $f_A : \{0, 1\}^\kappa \rightarrow \{0, 1\}^\omega$ by $f_A(x)(n) = x(A(n))$.

If $\mathcal{E} \subseteq \Sigma_\omega$ is dense in $\mathcal{A}_\omega$ and if $\mathcal{F} \subseteq [\kappa]^\omega$ is cofinal, then it is easy to check that $\{f_A^{-1}(E) : A \in \mathcal{F}, E \in \mathcal{E}\}$ is dense in $\mathcal{A}_\kappa$. This shows that $d(\mathcal{A}_\kappa) \leq d(\mathcal{A}_\omega) \cdot \text{cf}[\kappa]^\omega$. There remains to show that $\text{wd}(\mathcal{A}_\kappa) \geq d(\mathcal{A}_\omega) \cdot \text{cf}[\kappa]^\omega$.

Let $\mathcal{E}_\kappa \subseteq \Sigma_\kappa$ be a collection of Baire sets which is weakly dense in $\mathcal{A}_\kappa$. Define a family $\mathcal{E}_\omega \subseteq \Sigma_\omega$ as follows: If $E \in \mathcal{E}_\kappa$, let $C(E) \in \Sigma_\omega$ be a set of least measure satisfying $E \subseteq^* C(E) \times \{0, 1\}^{\kappa \setminus \omega}$. Let $\mathcal{E}_\omega = \{C(E) : E \in \mathcal{E}_\kappa\}$. Then $|\mathcal{E}_\omega| \leq |\mathcal{E}_\kappa|$ and $\mathcal{E}_\omega$ is weakly dense in $\mathcal{A}_\omega$. (*Proof.* If $F \in \Sigma_\omega, \mu F > 0$, then there is an $E \in \mathcal{E}_\kappa$ s.t. $E \subseteq^* F \times \{0, 1\}^{\kappa \setminus \omega}$ or $E \subseteq^* F^c \times \{0, 1\}^{\kappa \setminus \omega}$, say the former. By definition of $C(E)$, $C(E) \subseteq^* F$.) Thus $\text{wd}(\mathcal{A}_\kappa) \geq \text{wd}(\mathcal{A}_\omega)$. Now let $X = \{0, 1\}^\omega$ and identify $\{0, 1\}^\kappa$ with X^κ via the map $f \mapsto \langle f(\omega \cdot \alpha + n) : n \in \omega \rangle : \alpha < \kappa$. Now with $\mathcal{E}_\kappa \subseteq \Sigma_\kappa$ weakly dense in $\mathcal{A}_\kappa$, define a cofinal family $\mathcal{F} \subseteq [\kappa]^\omega$ by $\mathcal{F} = \{\text{supp}(E) : E \in \mathcal{E}_\kappa\}$ where $\text{supp}(E)$ is taken in X^κ . To see that $\mathcal{F}$ is cofinal, fix $B \in [\kappa]^\omega$. By Lemma 2, there is a measurable set $F \subseteq X^B$ s.t. for any measurable set $D \subseteq X^B$, if either $D \subseteq^* F$ or $D \subseteq^* F^c$ then $\text{supp}(D) = B$. Fix such an F and let $E \in \mathcal{E}_\kappa$ be s.t. $E \subseteq^* F \times X^{\kappa/B}$ or $E \subseteq^* F^c \times X^{\kappa/B}$, say the former. Claim: $B \subseteq \text{supp}(E)$. For suppose

$\alpha \in B \setminus \text{supp}(E)$. Then there is a countable $S \subseteq \kappa$ and an $\bar{E} \subseteq X^S$ such that $\alpha \notin S$ and $E = \bar{E} \times X^{\kappa \setminus S}$. By Lemma 1, there is a measurable set $D \subseteq^* X^{B \cap S}$ s.t.

$$E \subseteq^* D \times X^{\kappa \setminus (B \cap S)} \subseteq^* F \times X^{\kappa \setminus B}.$$

Hence $D \times X^{B \setminus S} \subseteq^* F$; by the choice of F , $B \setminus S = \emptyset$, i.e. $B \subseteq S$, contradiction. $\square$

As mentioned in the introduction we have

Corollary. For each $x \in \text{St}(\mathcal{A}_\kappa)$, $\pi\chi(x, \text{St}(\mathcal{A}_\kappa)) = \pi w(\text{St}(\mathcal{A}_\kappa))$. $\square$

(By a well-known theorem of D. Maharam [11a] every homogeneous measure algebra is isomorphic to $\mathcal{A}_\kappa$ for some κ .)

Remarks concerning the cardinal $cf[\kappa]^\omega$. (1) An easy induction on n shows that $cf[\aleph_n]^\omega = \aleph_n$ for $n \in \omega$.

(2) If either GCH or the covering lemma with respect to an inner model of GCH holds then $cf[\kappa]^\omega = \kappa$ if $cf(\kappa) > \omega$, κ^+ if $cf(\kappa) = \omega$. (See [9].)

(3) Assuming the consistency of large cardinals, there are models where $2^{\aleph_0} < \aleph_\omega$ and $\aleph_\omega^{\aleph_0} > \aleph_{\omega+1}$. In any such model $cf[\aleph_\omega]^\omega > \aleph_{\omega+1}$. (See [11].)

4. PARTIAL LIFTINGS

When liftings are constructed inductively (see e.g. [1, 4]) the problem arises of extending a partial lifting $\rho: B \rightarrow \Sigma_\omega$ to an element $a \in \mathcal{A}_\omega \setminus B$. In Theorem 2 below we compute the least size of a subalgebra B for which such an extension can fail to exist.

Let $\lambda = \min\{|\mathcal{C}|: \mathcal{C} \subseteq \mathcal{N}_\omega \text{ and } \bigcup \mathcal{C} \notin \mathcal{N}_\omega\}$.

Lemma 4. $\text{wd}(\mathcal{A}_\omega) \geq \min\{|A|: A \subseteq \{0, 1\}^\omega \text{ and } A \text{ has positive outer measure}\}$. In particular, $\text{wd}(\mathcal{A}_\omega) \geq \lambda$.

Proof. Let $D \subseteq \Sigma_\omega$ be dense in $\mathcal{A}_\omega$, $|D| = \text{wd}(\mathcal{A}_\omega)$.

For each $E \in D$, choose a point $x(E) \in E$ of density 1. Then $A = \{x(E): E \in D\}$ has outer measure 1. $\square$

Theorem 2. There is a subalgebra $B \subseteq \mathcal{A}_\omega$ of size λ as well as a strong partial lifting $\rho: B \rightarrow \Sigma_\omega$ and an element a of $\mathcal{A}_\omega \setminus B$ s.t. ρ cannot be extended to a partial lifting $\langle B \cup \{a\} \rangle \rightarrow \Sigma_\omega$.

Proof. Fix a measurable set $A \subseteq \{0, 1\}^\omega$ s.t. both A and A^c have a dense set of points of density 1 (see [13]). Let $a = [A]$. Also fix a collection $\mathcal{C} = \{N_\alpha: \alpha < \lambda\} \subseteq \mathcal{N}_\omega$ of subsets of A^c s.t. $\bigcup \mathcal{C} \notin \mathcal{N}_\omega$. Inductively construct an increasing sequence $\langle B_\alpha: \alpha < \lambda \rangle$ of subalgebras of $\mathcal{A}_\omega$, a sequence $\langle E_\alpha: \alpha < \lambda \rangle$ of elements of Σ_ω , and partial liftings $(\rho|_{B_\alpha}): B_\alpha \rightarrow \Sigma_\omega$ satisfying

- (0) For all α , $|B_\alpha| < \lambda$;
- (1) $B_0 = \{[E]: E \text{ is a clopen subset of } \{0, 1\}^\omega\}$; $\rho([E]) = E$ for all clopen E ;
- (2) $B_{\alpha+1} = \langle B_\alpha \cup \{e_\alpha\} \rangle$, $e_\alpha = [E_\alpha]$, and $\rho(e_\alpha) = E_\alpha \cup N_\alpha$;

(3) If α is a limit ordinal, $B_\alpha = \bigcup_{\beta < \alpha} B_\beta$, $\rho|_{B_\alpha} = \bigcup_{\beta < \alpha} \rho|_{B_\beta}$; and for all $\alpha < \lambda$ and $x \in B_\alpha$:

(4) If $x \neq 0$ then $x \wedge a \neq 0$;

(5) $x \leq a$ implies $\rho(x) \subseteq A \cup \bigcup_{\beta < \alpha} N_\beta$.

The initial and limit stages given by (1) and (3) present no difficulties ((4) and (5) hold at stage 0 by the choice of A).

Suppose now that $\rho|_{B_\alpha}$ is given. Since $|B_\alpha| < \lambda$, $B_\alpha|_a$ is not weakly dense in $\mathcal{A}_\omega|_a$. Hence there is an $e_\alpha = [E_\alpha] \in \mathcal{A}_\omega|_a (E_\alpha \subseteq A)$ s.t.

(6) For all $x \in B_\alpha$, if $x \neq 0$ then $e_\alpha \wedge x \wedge a \neq 0$ and $e_\alpha^c \wedge x \wedge a \neq 0$.

Extend ρ to $\langle B_\alpha \cup \{e_\alpha\} \rangle = B_{\alpha+1}$ by letting $\rho(e_\alpha) = E_\alpha \cup N_\alpha$ ((6) ensures that ρ extends: see [14, Theorem 12.2] for example). We have to check (4) and (5) for $\alpha + 1$.

Let $x \in B_{\alpha+1}$. Then for some $c, d \in B_\alpha$, $x = (e_\alpha \wedge c) \vee (d \setminus e_\alpha)$. If $x \neq 0$ then one of c, d is not 0. If $c \neq 0$ then by (6), $e_\alpha \wedge c \wedge a \neq 0$ and so $x \wedge a \neq 0$. Similarly if $d \neq 0$ then $x \wedge a \neq 0$ and so (4) holds. For (5), if $x \leq a$ then $d \leq a$ and $\rho(x) \subseteq \rho(e_\alpha \vee d) = \rho(e_\alpha) \cup \rho(d) = E_\alpha \cup N_\alpha \cup \rho(d) \subseteq A \cup \bigcup_{\beta < \alpha+1} N_\beta$. This completes the construction.

Now let $B = \bigcup_{\alpha < \lambda} B_\alpha$, $\rho = \bigcup_{\alpha < \lambda} \rho|_{B_\alpha}$. (1) ensures that ρ is a strong partial lifting. If ρ could be extended to a partial lifting $\langle B \cup \{a\} \rangle \rightarrow \Sigma_\omega$, we would have $\bigcup \mathcal{E} = \bigcup_{\alpha < \lambda} N_\alpha \subseteq \bigcup_{\alpha < \lambda} \rho(e_\alpha) \subseteq \rho(a)$ but $\bigcup \mathcal{E}$ is a nonnull subset of A^c . $\square$

Remarks. (1) If $|B| < \lambda$ then ρ extends: simply let $\rho(a) = A \cup \bigcup \{\rho(b) : b \leq a \text{ and } b \in B\} \setminus \bigcup \{\rho(b) : b \leq a^c \text{ and } b \in B\}$.

(2) A similar argument can be used to get a subalgebra $B \subseteq \mathcal{A}_\omega$ of size $\aleph_1$ and a partial strong Borel lifting $\rho : B \rightarrow \Sigma_\omega$ (i.e. $\rho(a)$ is Borel for each $a \in B$) which does not extend to a partial Borel lifting $\langle B \cup \{a\} \rangle \rightarrow \Sigma_\omega$.

(3) If we replace the condition that ρ is strong by the stronger condition that $(\forall [E] \in B)(\rho(E) \supseteq \{x \in \{0, 1\}^\omega : d(x, E) = 1\})$ then ρ extends to a lifting $\mathcal{A}_\omega \rightarrow \Sigma_\omega$ regardless of the size of B : for each $x \in \{0, 1\}^\omega$ let $\mathcal{F}_x$ be an ultrafilter extending $\{[E] : d(x, E) = 1\} \cup \{\rho(b) : b \in B, x \in \rho(b)\}$. Define $\rho(a) = \{x \in \{0, 1\}^\omega : a \in \mathcal{F}_x\}$ for all $a \in \mathcal{A}_\omega$. It is straightforward to check that this defines a lifting extending ρ .

Remark. W. Just has shown that GCH implies that the answer to the question in the introduction is no for ccc algebras. Also W. Weiss has shown that, in the notation of (5) of the remarks in the introduction, there is a dense set of points $x \in \text{St}(B)$ such that $\text{wd}(B) = \pi\chi(x, \text{St}(B))$. Thus the weak density of B is the minimum π -character of a point in $\text{St}(B)$.

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