ON STRONG UNIQUENESS IN ONE-SIDED L^1 -APPROXIMATION OF DIFFERENTIABLE FUNCTIONS

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ABSTRACT. We consider the problem of one-sided L^1 -approximation in $C^1[0, 1]$. A sharp estimate for the rate of strong uniqueness for arbitrary unicity subspaces of $C^1[0, 1]$ is given.

INTRODUCTION

Let $U_n = \text{span}[p_1, \dots, p_n]$ be an n -dimensional subspace of the space $C[0, 1]$ of real continuous functions. Then $u_f \in U_n$ is called a best one-sided L^1 -approximation of $f \in C[0, 1]$ if $u_f \leq f$ on $[0, 1]$ and $\|f - u_f\| \leq \|f - u\|$ for all $u \in U_n$ satisfying $u \leq f$ on $[0, 1]$. (Here $\|\cdot\|$ denotes the usual L^1 -norm on $[0, 1]$). In order to ensure existence of best approximants we shall assume throughout the paper without further reference that U_n contains a *strictly positive* function on $[0, 1]$.

It is well known that for the uniqueness of best one-sided L^1 -approximants it is necessary to restrict the consideration to the space $C^1[0, 1]$ of continuously differentiable functions (see [4] for details.) We shall say that $U_n \subset C^1[0, 1]$ is a unicity subspace of $C^1[0, 1]$ if every $f \in C^1[0, 1]$ has a unique best one-sided L^1 -approximant in U_n .

Let U_n be a unicity subspace of $C^1[0, 1]$ and assume that O is the best approximant of a given $f \in C^1[0, 1]$. (This can be always achieved by subtracting from a given function its best approximant.) Then the best approximant is called strongly unique of order γ ($\gamma \geq 1$) if for all $u \in U_n$ such that $u \leq f$ on $[0, 1]$ and $\|f - u\| \leq \|f\| + 1$ we have

$$(1) \quad \|f - u\| \geq \|f\| + K_f \|u\|^\gamma,$$

where $K_f > 0$ depends only on f and U_n . (The reason why we consider only u 's $\in U_n$ satisfying $\|f - u\| \leq \|f\| + 1$ consists in the fact that (1) does not

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hold if $\gamma > 1$ and $\|u\| \rightarrow \infty$.) If $\gamma = 1$ then we simply say that the best approximant is strongly unique.

More generally, the problem of strong uniqueness consists in finding a positive continuous function $\varphi_f(\delta) \rightarrow 0$, $\delta \rightarrow +0$ such that

$$\|f - u\| \geq \|f\| + \|u\|\varphi_f(\|u\|)$$

for every $u \in U_n$ satisfying $u \leq f$ and $\|f - u\| \leq \|f\| + 1$. In this note we shall solve the above problem by determining the exact rate of $\varphi_f(\delta)$ which, in general, depends on f . (For two-sided L^1 -approximation this is done in Kroó [2].)

MAIN RESULTS

Let $U_n = \text{span}[p_1, \dots, p_n]$ be a unicity subspace of $C^1[0, 1]$. For $f \in C^1[0, 1]$ set $H_f(\delta) = \omega(f', \delta) + \sum_{i=1}^n \omega(p'_i, \delta) + \delta$, where

$$\omega(g, \delta) = \max\{|g(x_1) - g(x_2)| : x_1, x_2 \in [0, 1], |x_1 - x_2| \leq \delta\}$$

is the usual modulus of continuity of $g \in C[0, 1]$. In what follows $H_f^{-1}(\delta)$ is the inverse function of $H_f(\delta)$.

Theorem 1. *Let $U_n = \text{span}[p_1, \dots, p_n]$ be a unicity subspace of $C^1[0, 1]$ and consider an $f \in C^1[0, 1]$ with O being its best one-sided L^1 -approximant. Then for every $u \in U_n$ such that $u \leq f$ on $[0, 1]$ and $\|f - u\| \leq \|f\| + 1$ we have*

$$(2) \quad \|f - u\| \geq \|f\| + c_1 \|u\| H_f^{-1}(c_2 \|u\|),$$

where $c_1, c_2 > 0$ depend only on f and U_n .

If, in particular, $H_f(\delta) = O(\delta^\alpha)$, $0 < \alpha \leq 1$, i.e. f' and derivatives of elements of U_n are $\text{Lip } \alpha$ functions, then it follows from (2) that strong uniqueness or order $(\alpha + 1)/\alpha$ takes place.

Our second statement shows that the order of strong uniqueness given above is, in general, the best possible.

Theorem 2. *Let $U_n (n \geq 2)$ be a unicity subspace of $C^1[0, 1]$. Then there exists a $g \in C^1[0, 1]$ with O being its best one-sided L^1 -approximant and a sequence of functions $u_k \in U_n$, $u_k \leq g$ ($k = 1, 2, \dots$), $u_k \rightarrow 0$ ($k \rightarrow \infty$) such that for some $c_3, c_4 > 0$ independent of k*

$$(3) \quad \|g - u_k\| \leq \|g\| + c_3 \|u_k\| H_g^{-1}(c_4 \|u_k\|), \quad k \geq 1.$$

PRELIMINARY RESULTS

We shall need several known results from the theory of one-sided L^1 -approximation. For $f \in C^1[0, 1]$ set $Z_1(f) = \{x \in [0, 1] : f(x) = 0, \text{ and } f'(x) = 0 \text{ if } x \in (0, 1)\}$.

Proposition 1 [1, p. 80–82]. Let $U_n \subset C^1[0, 1]$, $f \in C^1[0, 1]$. In order that O be a best one-sided L^1 -approximant of f in U_n it is necessary and sufficient that $f \geq 0$ and there exist points $x_1, \dots, x_k \in Z_1(f)$ ($1 \leq k \leq n$) and positive numbers $\lambda_1, \dots, \lambda_k$ such that

$$(4) \quad \int_0^1 u = \sum_{i=1}^k \lambda_i u(x_i), \quad u \in U_n.$$

Proposition 2 [4]. Let $U_n \subset C^1[0, 1]$. Then the following statements are equivalent:

- (i) U_n is a unicity subspace of $C^1[0, 1]$;
- (ii) there do not exist a $u^* \in U_n \setminus \{0\}$, points $\{x_i\}_{i=1}^k$ in $Z_1(u^*)$ ($1 \leq k \leq n$) and positive numbers $\{\lambda_i\}_{i=1}^k$ so that (4) holds for all $u \in U_n$.

The next proposition shows the existence of positive quadrature formulas with less than n active points.

Proposition 3 [3]. Let U_n be an n -dimensional subspace of $C[0, 1]$ ($n \geq 2$) containing a strictly positive function. Then there exist points $x_1, \dots, x_k \in [0, 1]$, $1 \leq k \leq n-1$, and positive numbers $\lambda_1, \dots, \lambda_k$ so that (4) holds for every $u \in U_n$.

Proof of Theorem 1. Let U_n, f and $u \in U_n \setminus \{0\}$ satisfy the assumptions of the theorem and set $\varepsilon = \|f - u\| - \|f\|$; then $0 < \varepsilon \leq 1$. Since O is the best one-sided L_1 -approximant of f by Proposition 1 there exist points $\{x_i\}_{i=1}^k$ in $Z_1(f)$, $1 \leq k \leq n$, and positive numbers $\{\lambda_i\}_{i=1}^k$ such that (4) holds for every element of U_n . In particular,

$$(5) \quad \varepsilon = \|f - u\| - \|f\| = \int_0^1 (f - u) - \int_0^1 f = - \int_0^1 u = - \sum_{i=1}^k \lambda_i u(x_i).$$

Furthermore, $u(x_i) \leq f(x_i) = 0$, $1 \leq i \leq k$, hence applying this in (5) we obtain

$$(6) \quad |u(x_i)| \leq a_1 \varepsilon, \quad 1 \leq i \leq k.$$

(Here and in what follows $a_1, a_2, \dots$, denote positive constants depending only on f and U_n). Using that $\|u\| \leq \|f - u\| + \|f\| \leq 2\|f\| + 1$ and all norms are equivalent in U_n we have, representing u in the form $u = \sum_{i=1}^n t_i p_i$ ($t_i \in \mathbb{R}$),

$$(7) \quad \begin{aligned} \omega(u', \delta) &\leq \sum_{i=1}^n |t_i| \omega(p_i', \delta) \leq \max_{1 \leq i \leq n} |t_i| \sum_{i=1}^n \omega(p_i', \delta) \\ &\leq a_2 \|u\| H_f(\delta) \leq a_3 H_f(\delta). \end{aligned}$$

Assume now that $x_i \in (0, 1)$ for some $1 \leq i \leq k$. Let $\varepsilon_0 = \min_{0 \leq i \leq k} (x_{i+1} - x_i)$ ($x_0 = 0, x_{k+1} = 1$), and choose an arbitrary $h \in \mathbb{R}$, $0 < h < \varepsilon_0$. Then setting $\xi = \operatorname{sgn} u'(x_i)$ we have

$$u(x_i + \xi h) = u(x_i) + h|u'(x_i)| + \rho_1,$$

where by (7) $|\rho_1| \leq h\omega(u', h) \leq a_3 h H_f(h)$. Moreover, since $f(x_i) = f'(x_i) = 0$ we also have $|f(x_i + \xi h)| \leq h\omega(f', h) \leq h H_f(h)$. Thus applying that $u(x_i + \xi h) \leq f(x_i + \xi h)$ it follows by (6) that

$$(8) \quad \begin{aligned} |u'(x_i)| &= \frac{1}{h}(u(x_i + \xi h) - u(x_i) - \rho_1) \\ &\leq \frac{1}{h}(f(x_i + \xi h) + a_1 \varepsilon - \rho_1) \leq a_4 \left(H_f(h) + \frac{\varepsilon}{h} \right), \end{aligned}$$

where $x_i \in (0, 1)$ and $0 < h < \varepsilon_0$ are arbitrary.

Consider now the functional $L: U_n \rightarrow \mathbf{R}$ defined by

$$L(\tilde{u}) = \sum_{i=1}^k |\tilde{u}(x_i)| + \sum_{x_i \in (0, 1)} |\tilde{u}'(x_i)|, \quad \tilde{u} \in U_n.$$

By Proposition 2 $L(\tilde{u}) = 0$ implies $\tilde{u} \equiv 0$. Thus L is a norm on U_n , hence the equivalence of norms on U_n yields that

$$(9) \quad \|\tilde{u}\| \leq a_5 L(\tilde{u}), \quad \tilde{u} \in U_n.$$

Applying now (6) and (8) in (9) we easily obtain

$$(10) \quad \|u\| \leq a_6(\varepsilon/h + H_f(h)).$$

We shall distinguish now two cases.

Case 1. $\|u\| > 2a_6\varepsilon/\varepsilon_0$. Then we may set $h = 2a_6\varepsilon/\|u\| < \varepsilon_0$ in (10) implying

$$\frac{\|u\|}{a_6} \leq \frac{\|u\|}{2a_6} + H_f(2a_6\varepsilon/\|u\|).$$

Hence $a_7\|u\| \leq H_f(2a_6\varepsilon/\|u\|)$ and therefore $2a_6\varepsilon \geq \|u\|H_f^{-1}(a_7\|u\|)$ verifying the theorem in this case.

Case 2. $\|u\| \leq 2a_6\varepsilon/\varepsilon_0$. In this case (2) is trivial because

$$H_f^{-1}(\|u\|) \leq H_f^{-1}(2a_6\varepsilon/\varepsilon_0) \leq a_8.$$

The proof of Theorem 1 is completed.

Proof of Theorem 2. Set $\psi(t) = \sum_{i=1}^n \omega(p'_i, t) + t$, $\varphi(x) = \int_0^x \psi(t) dt$, $0 \leq x \leq 1$.

By Proposition 3 there exist points $x_0 = 0 \leq x_1 < \dots < x_{k_0} \leq 1 = x_{k_0+1}$, $1 \leq k_0 \leq n-1$, and positive numbers $\{\lambda_i\}_{i=1}^{k_0}$ such that (4) holds.

Consider the function φ_h on $[0, 4h]$ ($0 < h \leq 1$) defined as follows:

$$\varphi_h(x) = \begin{cases} \varphi(x), & x \in [0, h], \\ 2\varphi(h) - \varphi(2h - x), & x \in [h, 2h], \end{cases}$$

and

$$\varphi_h(x) = \varphi_h(4h - x) \quad \text{for } x \in [2h, 4h].$$

Then $\varphi_h \in C^1[0, 4h]$; $\varphi_h > 0$ on $(0, 4h)$; $\varphi_h(0) = \varphi'_h(0) = \varphi_h(4h) = \varphi'_h(4h) = 0$ and $b_2\psi(\delta) \leq \omega(\varphi'_h, \delta) \leq b_1\psi(\delta)$ ($b_1, b_2 > 0, \delta \in [0, 4h]$). Now we in-

roduce the function g by $g(x) = \varphi_{h_i}(x - x_i)$, where $h_i = (x_{i+1} - x_i)/4$, $x \in [x_i, x_{i+1}]$, $0 \leq i \leq k_0$. Then $g \in C^1[0, 1]$; $x_1, \dots, x_{k_0} \in Z_1(g)$; $g > 0$ if $x \in [0, 1]$, $x \neq x_i$, $0 \leq i \leq k_0 + 1$ and $b_4\psi(\delta) \leq \omega(g', \delta) \leq b_3\psi(\delta)$. By Proposition 1 O is the best one-sided L^1 -approximant of g . Since $1 \leq k_0 \leq n - 1$, there exists a $p \in U_n \setminus \{0\}$ such that $p(x_i) = 0$, $1 \leq i \leq k_0$. Set

$$(11) \quad M_k = \max_{0 \leq x \leq 1} \left(\frac{1}{k} p(x) - g(x) \right) = \frac{1}{k} p(\xi_k) - g(\xi_k), \quad 0 \leq \xi_k \leq 1.$$

Proposition 2 implies that among x_i 's, $1 \leq i \leq k_0$, there should be at least one $x_j \in (0, 1)$ for which $p'(x_j) \neq 0$. Since $x_i \in Z_1(g)$, $1 \leq i \leq k_0$, it follows that $M_k > 0$ for every $k = 1, 2, \dots$. Furthermore, $\min_{0 \leq i \leq k_0+1} |\xi_k - x_i| \rightarrow 0$ as $k \rightarrow \infty$. Hence setting $\min_{0 \leq i \leq k_0+1} |\xi_k - x_i| = |\xi_k - x_s|$ ($0 \leq s \leq k_0 + 1$) we have for k large enough

$$\begin{aligned} M_k &= \frac{1}{k} p(\xi_k) - g(\xi_k) = \frac{1}{k} \{p(\xi_k) - p(x_s)\} - \varphi(|\xi_k - x_s|) \\ &\leq b_5(|\xi_k - x_s|/k - \varphi(|\xi_k - x_s|)) \leq b_5 \max_{0 \leq h \leq 1} (h/k - \varphi(h)) \\ &\leq b_5(y_0/k - \varphi(y_0)), \end{aligned}$$

where $\varphi'(y_0) = \psi(y_0) = 1/k$. Using that $H_g(\delta) \leq b_6\psi(\delta)$ we have $y_0 \leq H_g^{-1}(b_6/k)$, i.e.

$$(12) \quad M_k \leq b_5 y_0/k \leq b_5 H_g^{-1}(b_6/k)/k.$$

Let $u_0 \in U_n$ be positive on $[0, 1]$, $u_0(x) \geq m_0 > 0$ for $x \in [0, 1]$. Set for $k \in \mathbb{N}$

$$(13) \quad u_k(x) = \frac{1}{k} p(x) - \frac{b_5 H_g^{-1}(b_6/k)}{k m_0} u_0(x) \in U_n.$$

Then $u_k \rightarrow 0$ ($k \rightarrow \infty$) and, in addition, by (12) and (11) $u_k(x) \leq (1/k)p(x) - M_k \leq g(x)$, $x \in [0, 1]$. Moreover, by (4) $\int_0^1 p = \sum_{i=1}^{k_0} \lambda_i p(x_i) = 0$, hence

$$\begin{aligned} (14) \quad \|g - u_k\| - \|g\| &= - \int_0^1 u_k = - \frac{1}{k} \int_0^1 p + b_7 \frac{H_g^{-1}(b_6/k)}{k} \\ &= b_7 \frac{H_g^{-1}(b_6/k)}{k}. \end{aligned}$$

Furthermore (13) implies that $u_k(x) = p(x)/k + o(1/k)$ that is $\|u_k\| \geq b_8/k$ for k large enough. This and (14) yields

$$\begin{aligned} \|g - u_k\| &= \|g\| + b_7 \frac{H_g^{-1}(b_6/k)}{k} \\ &\leq \|g\| + b_9 \|u_k\| H_g^{-1}(b_{10} \|u_k\|). \end{aligned}$$

Thus g and u_k ($k = 1, 2, \dots$) satisfy all requirements of Theorem 2.

Let us conclude this note with several remarks.

Assume that O is the best one-sided L^1 -approximation of $f \in C^1[0, 1]$, $x_1, \dots, x_n \in Z_1(f)$ are as in Proposition 1 (with $k = n$) and

$$\det\{p_j(x_i)\}_{i,j=1}^n \neq 0.$$

Then we can easily derive from (6) that $\|u\| \leq c\varepsilon = c\{\|f - u\| - \|f\|\}$, that is O is a strongly unique best approximant. (It is shown in [1] that in order to have strong uniqueness it is necessary that $Z_1(f)$ contains at least n distinct points.) Finally, let us mention that the rate of strong uniqueness for a given $f \in C^1[0, 1]$ with O being its best approximant can be better if for some $r \geq 2$ $f^{(j)}(x) = 0$, $2 \leq j \leq r$, whenever $x \in Z_1(f)$. (This can be proved analogously to Theorem 1).

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