MANIFOLDS WITH $(\mathbb{Z}_2)^k$ -ACTION

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ABSTRACT. Let $(M^n, T_1, \dots, T_k)$ be a closed manifold with differentiable $(\mathbb{Z}_2)^k$ -action. The purpose of this paper is to show how to construct the bordism class $[M^n]$ in terms of the decomposed form of the fixed point data of this action.

1. INTRODUCTION

In their book "Differentiable Periodic Maps", P. E. Conner and E. E. Floyd proved that the equivariant bordism class of a closed manifold with involution (M^n, T) is completely determined by the fixed data (F, ν) , where F is the fixed point set and ν is its normal bundle. In particular, M is bordant to $\mathbb{RP}(\nu \oplus 1)$.

For actions of $(\mathbb{Z}_2)^k$, it is shown in [2] that the equivariant bordism class is also determined by the fixed data. In this case, the fixed data consists of the fixed point set and the subbundles of the normal bundle on which $(\mathbb{Z}_2)^k$ has given representations.

The purpose of this note is to show how to construct the bordism class of the original manifold in terms of that fixed data. Specifically, the main result is

Proposition. *Being given a closed manifold M^n with commuting involutions $T_1, T_2, \dots, T_k$, there is a sequence $(M_i^n, T_1, \dots, T_i)$, $0 \leq i \leq k$, of closed manifolds with $(\mathbb{Z}_2)^i$ -action for which*

- (1) $(M_k^n, T_1, \dots, T_k) = (M^n, T_1, \dots, T_k)$,
- (2) the fixed data of $(\mathbb{Z}_2)^{i-1}$ acting on M_{i-1}^n is explicitly constructible from the fixed data of $(\mathbb{Z}_2)^i$ acting on M_i^n , and
- (3) M_{i-1}^n is bordant to M_i^n .

In particular, M^n is bordant to M_0^n , which is constructible from the fixed data of the original $(\mathbb{Z}_2)^k$ -action.

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2. PRELIMINARIES

In order to describe actions of $(\mathbb{Z}_2)^k$ one needs a certain amount of notation. One needs a certain amount of complication, but wants to avoid excess messiness.

Throughout this description, $(\mathbb{Z}_2)^k$ will always be considered as the group generated by k commuting involutions $T_1, T_2, \dots, T_k$.

Being given a closed n -dimensional manifold M^n with $(\mathbb{Z}_2)^k$ -action the fixed set of the action is a closed submanifold $F \subset M$ and the normal bundle of F in M , ν , decomposes as a Whitney sum of subbundles on which $(\mathbb{Z}_2)^k$ acts as one of the irreducible (nontrivial) real representations.

To describe the irreducible representations of $(\mathbb{Z}_2)^k$, one may use sequences $a = (a_1, a_2, \dots, a_k)$ where each a_j is either 0 or 1. For the given sequence, the corresponding representation is the real numbers, $\mathbb{R}$, on which T_j acts as multiplication by $(-1)^{a_j}$ for each j .

If one denotes by $\varepsilon_a \subset \nu$ the subbundle on which each T_j acts as $(-1)^{a_j}$, then

$$\nu = \bigoplus_{a \neq (0)} \varepsilon_a$$

where the sum excludes the trivial representation given by $(0, 0, \dots, 0)$. Thus, the fixed data of the action on M is given by F and the $2^k - 1$ vector bundles ε_a , $a \neq (0)$.

3. THE CONSTRUCTION

To construct the sequence of actions of $(\mathbb{Z}_2)^i$ on M_i^n , one proceeds by downward induction on i . It then suffices to give the description for construction M_{k-1}^n from $M^n = M_k^n$.

So let M^n be given with involutions $T_1, T_2, \dots, T_k$. Let F_{T_k} denote the fixed point set of the involution T_k acting on M and let μ denote its normal bundle in M . One lets M_{k-1}^n be the manifold $\mathbf{RP}(\mu \oplus 1)$.

On F_{T_k} one has an induced action of $(\mathbb{Z}_2)^{k-1}$, the group generated by $T_1, T_2, \dots, T_{k-1}$. This group also acts on μ via bundle maps, and letting $(\mathbb{Z}_2)^{k-1}$ act on $F_{T_k} \times \mathbb{R}$ by its action on F_{T_k} and trivial action in $\mathbb{R}$ one obtains an action on the trivial bundle 1. These actions induce an action of $(\mathbb{Z}_2)^{k-1}$ on $\mathbf{RP}(\mu \oplus 1)$ which is $(M_{k-1}^n, T_1, \dots, T_{k-1})$.

It is clear that $(M_k^n, T_1, \dots, T_k) = (M^n, T_1, \dots, T_k)$ since the construction works down from $(M^n, T_1, \dots, T_k)$. It is also clear that M_{i-1}^n is bordant to M_i^n by the result of Conner and Floyd, for M_i^n is bordant to $\mathbf{RP}(\mu \oplus 1)$ over F_{T_i} , where μ is the normal bundle.

The only remaining point is to compute the fixed data of $(\mathbb{Z}_2)^{k-1}$ acting on M_{k-1}^n and to see that it depends only on the fixed data of $(\mathbb{Z}_2)^k$ acting on

M_k^n . The final point of the Proposition, that M_0^n is constructible, is simply the observation that M_0^n is the fixed set of the $(\mathbb{Z}_2)^0 = \{1\}$ -action.

To find the fixed data of $(\mathbb{Z}_2)^{k-1}$ on $\mathbf{RP}(\mu \oplus 1)$, one first observes that $\mathbf{RP}(\mu \oplus 1)$ fibers equivariantly over F_{T_k} , so that the fixed set of $(\mathbb{Z}_2)^{k-1}$ on $\mathbf{RP}(\mu \oplus 1)$ is mapped into that on F_{T_k} , which is F (the fixed set of $(\mathbb{Z}_2)^k$ on M).

Thus the fixed set of $(\mathbb{Z}_2)^{k-1}$ on $\mathbf{RP}(\mu \oplus 1)$ is the fixed set of $(\mathbb{Z}_2)^{k-1}$ on $\mathbf{RP}(\mu|_F \oplus 1) = \mathbf{RP}((\oplus_a \varepsilon_{(a,1)}) \oplus 1)$, where $\mu|_F = \oplus_a \varepsilon_{(a,1)}$ is the restriction of μ to F .

It is immediate that the fixed set is then given by $\mathbf{RP}(\varepsilon_{(0,1)} \oplus 1)$ and the $\mathbf{RP}(\varepsilon_{(a,1)})$ with $a \neq (0, \dots, 0)$.

To describe the normal bundle to the fixed set, one notes that the normal bundle of F in F_{T_k} is $\oplus_{a \neq (0)} \varepsilon_{(a,0)}$ and that the normal bundle of $\mathbf{RP}(\varepsilon_{(0,1)} \oplus 1)$ in $\mathbf{RP}(\mu|_F \oplus 1)$ is $\lambda \otimes (\oplus_{a \neq (0)} \varepsilon_{(a,1)})$ and that of $\mathbf{RP}(\varepsilon_{(a,1)})$ in $\mathbf{RP}(\mu|_F \oplus 1)$ is $\lambda \otimes ((\oplus_{b \neq a} \varepsilon_{(b,1)}) \oplus 1)$ where λ denotes the usual line bundle over a real projective space bundle consisting of vectors in the given line.

Over $\mathbf{RP}(\varepsilon_{(0,1)} \oplus 1)$, $(\mathbb{Z}_2)^{k-1}$ acts trivially in the fibers of λ , and over $\mathbf{RP}(\varepsilon_{(a,1)})$ it acts as (a) in λ . Hence, one can read off the decomposition of the normal bundle via representations.

If $c = (c_1, \dots, c_{k-1})$ gives a representation for $(\mathbb{Z}_2)^{k-1}$, the fixed data of $(M_{k-1}^n, T_1, \dots, T_{k-1})$ is given by $\mathbf{RP}(\varepsilon_{(0,1)} \oplus 1)$ with $\varepsilon_c = (\lambda \otimes \varepsilon_{(c,1)}) \oplus \varepsilon_{(c,0)}$ (where the $\varepsilon_{(c,x)}$ bundles are pulled back from F) and by the $\mathbf{RP}(\varepsilon_{(a,1)})$ with

$$\varepsilon_c = \begin{cases} (\lambda \otimes (\varepsilon_{(0,1)} \oplus 1)) \oplus \varepsilon_{(a,0)} & \text{if } c = a, \\ (\lambda \otimes \varepsilon_{(a+c,1)}) \oplus \varepsilon_{(c,0)} & \text{if } c \neq a, \end{cases}$$

(where for $a = (a_1, \dots, a_{k-1})$, $a + c = (a_1 + c_1, \dots, a_{k-1} + c_{k-1})$ with $a_j + c_j$ being 0 or 1, taken modulo 2).

This completes the description of the fixed data of $(\mathbb{Z}_2)^{k-1}$ on M_{k-1}^n . Similar descriptions for the fixed data of $(\mathbb{Z}_2)^i$ on M_i^n require iterative calculation. They are obviously unpleasant projective space constructions, but no theoretical difficulty is involved.

4. THE SPECIAL CASE $k = 2$

While the general case is obviously an unpleasant mess, the case $k = 2$ is quite simple. Being given (M^n, T_1, T_2) , with fixed data $(F, \varepsilon_{(1,0)}, \varepsilon_{(0,1)}, \varepsilon_{(1,1)})$, one can write down the class of M^n . Specifically one has

Corollary. *For the $(\mathbb{Z}_2)^2$ -action (M^n, T_1, T_2) , the manifold M^n is bordant to the union of*

$$\mathbf{RP}((\lambda \otimes \varepsilon_{(1,1)}) \oplus \varepsilon_{(1,0)} \oplus 1)$$

and

$$\mathbf{RP}((\lambda \otimes \varepsilon_{(0,1)}) \oplus \lambda \oplus \varepsilon_{(1,0)} \oplus 1)$$

where the first fibers over $\mathbf{RP}(\varepsilon_{(0,1)} \oplus 1)$ and then over F and the second fibers over $\mathbf{RP}(\varepsilon_{(1,1)})$ and then over F .

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