ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO

$$\Delta u + Ku^\sigma = 0 \text{ ON } R^n \text{ FOR } n \geq 3$$

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ABSTRACT. The equation $\Delta u + Ku^\sigma = 0$ is considered in R^n for $n \geq 3$, K a Hölder continuous function and σ a positive constant. If $K = O(|x|^{-l})$ for $l > 2$, we determine the asymptotic behavior of bounded solutions. In the case K is nonpositive and σ is greater than one, we show that the first term in the asymptotic description may be chosen arbitrarily.

1. INTRODUCTION

Let K be a Hölder continuous function, not identically zero, on R^n , $n \geq 3$, and σ a positive constant greater than 1. Ni [9] has shown that if $K = O(|x|^{-l})$ as $|x| \rightarrow \infty$ for $l > 2$, then the equation

$$(A) \quad \Delta u + Ku^\sigma = 0$$

has an infinite number of bounded positive solutions that are bounded away from 0.

For (A) on all of R^n , Ni [9] shows that for $l > 2$, if K doesn't change sign then a bounded solution u approaches a constant u_∞ as $|x| \rightarrow \infty$. Naito [8] shows that the sign condition on K is unnecessary and that the constant can be specified arbitrarily in an interval determined by K . In [5], Li and Ni show that the next term in the asymptotic description of u is given by

$$(B) \quad |u - u_\infty| = \begin{cases} O(|x|^{2-n}) & \text{if } l > n; \\ O(|x|^{2-n} \log |x|) & \text{if } l = n; \\ O(|x|^{2-l}) & \text{if } 2 < l < n. \end{cases}$$

The first result in this paper gives a more complete asymptotic description of solutions to (A) on R^n in the case that $l > n$. Here we may take any $\sigma > 0$.

Theorem 1. Suppose $K = O(|x|^{-l})$ as $|x| \rightarrow \infty$ for some $l > 2$ and $\sigma > 0$. If u is a bounded positive solution to $\Delta u + Ku^\sigma = 0$ then

- (i) there is a constant u_∞ such that $u - u_\infty = O(|x|^\gamma)$ as $|x| \rightarrow \infty$ for all $\gamma > \max\{2 - l, 2 - n\}$

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- (ii) if $l > n + m$ then there are $m + 1$ homogeneous harmonic polynomials $f_0, f_1, \dots, f_m$ where f_j is of degree j such that

$$u - u_\infty - \frac{f_0}{|x|^{n-2}} - \dots - \frac{f_m}{|x|^{n-2+2m}} = O(|x|^\gamma) \quad \text{as } |x| \rightarrow \infty$$

for all $\gamma > \max\{2 - l, 1 - n - m\}$.

Therefore, if K has compact support or decays exponentially as $|x| \rightarrow \infty$ we get a full asymptotic expansion of u .

In the case $K \leq 0$, we improve Naito's result to show that the first term in the asymptotic description can be chosen arbitrarily.

Theorem 2. Suppose $K = O(|x|^{-l})$ with $l > 2$ and $K \leq 0$ on R^n and $\sigma > 1$. Given any $c > 0$, there exists a unique positive solution u to $\Delta u + Ku^\sigma = 0$ with $u - c = O(|x|^\gamma)$ as $|x| \rightarrow \infty$ for all $\gamma > \max\{2 - l, 2 - n\}$.

The techniques used to prove both theorems involve knowledge of how the Laplacian acts on weighted Sobolev spaces as discussed in the next section.

Remarks. 1. All of the above results require $l > 2$. In the case $K \leq 0$ and $|K| \geq C|x|^{-l}$ as $|x| \rightarrow \infty$ for some $l \leq 2$, Ni shows that there are no positive solutions. Li and Ni [5] have some results for $K \geq 0$ and $0 < l < 2$ for the radial case.

2. It is interesting to compare the asymptotics (B) with the results of Meyers in [7]. Meyers studies the Poisson equation on an exterior domain. He has shown that if $\Delta u = f$ where $f = O(|x|^{-l})$ as $|x| \rightarrow \infty$ then there is a solution u with

$$u = \begin{cases} O(|x|^{2-l}) & \text{if } 2 < l < n \text{ or } l \text{ is not an integer;} \\ O(|x|^{2-l} \log |x|) & \text{otherwise.} \end{cases}$$

2. PRELIMINARIES

The weighted space $W_{s,\delta}^p$, for $1 < p < \infty$, s a nonnegative integer and δ real, is defined to be the closure of $C_0^\infty(R^n)$ in the norm

$$\|u\|_{p,s,\delta} = \sum_{|\alpha| \leq s} \|(1 + |x|^2)^{(\delta+|\alpha|)/2} D^\alpha u\|_{L^p}.$$

If $s = 0$ then $W_{0,\delta}^p = L_\delta^p$.

We use the following results about these spaces. These results can be found in McOwen [6], Cantor [2], and Nirenberg and Walker [10].

- (1) $W_{s,\delta}^p \subset W_{s',\delta'}^p$ if $s \geq s'$ and $\delta \geq \delta'$. The injection is compact if $s > s'$ and $\delta > \delta'$.
- (2) For $s > n/p$, if $f \in W_{s,\delta}^p$ then $f = O(|x|^\gamma)$ as $|x| \rightarrow \infty$ for all $\gamma > -n/p - \delta$.

(3) If $-n/p < \delta < -n/p + (n-2)$ then there is a constant A such that

$$\|u\|_{p,2,\delta} \leq A \|\Delta u\|_{L_{\delta+2}^p}.$$

(4) $W_{s,\delta}^p \subset \{ \text{continuous functions that approach 0 as } |x| \rightarrow \infty \}$ provided $s > n/p$ and $\delta > -n/p$.

(5) $\Delta: W_{2,\delta}^p \rightarrow L_{\delta+2}^p$ is an isomorphism for $-n/p < \delta < (n-2) - n/p$.

(6) $\Delta: W_{2,\delta}^p \rightarrow L_{\delta+2}^p$ is a surjection for $-n/p - m - 1 < \delta < -n/p - m$ with nullspace

$$N_m = \bigcup_{j=0}^m \mathcal{H}_j$$

where $\mathcal{H}_j = \{ \text{homogeneous harmonic polynomials of degree } j \}$.

(7) $\Delta: W_{2,\delta}^p \rightarrow L_{\delta+2}^p$ is an injection for $(n-2) - n/p + m < \delta < (n-2) - n/p + m + 1$ with range

$$\mathcal{R} = \left\{ g \in L_{\delta+2}^p : \int_{R^n} g(y) H(y) dy = 0 \quad \text{for all } H \in \bigcup_{j=0}^m \mathcal{H}_j \right\}.$$

3. PROOF OF THEOREM 1

Suppose u is a bounded positive solution to $\Delta u + Ku^\sigma = 0$. Choose $p > n$. Let $f = -Ku^\sigma$. Since u is bounded, $f \in L_{\delta+2}^p$ for all $\delta < (l-2) - n/p$. Since $l > 2$, there is a δ satisfying

$$-\frac{n}{p} < \delta < \min\left\{(l-2) - \frac{n}{p}, (n-2) - \frac{n}{p}\right\}.$$

For such a δ , $\Delta: W_{2,\delta}^p \rightarrow L_{\delta+2}^p$ is an isomorphism. Since $f \in L_{\delta+2}^p$, there is $v \in W_{2,\delta}^p$ with $\Delta v = f$. By (4), v is continuous and approaches 0 as $|x| \rightarrow \infty$. Since $v \in W_{2,\delta}^p$, $v = O(|x|^\gamma)$ for all $\gamma > -n/p - \delta$. This is true for any choice of δ with $-n/p < \delta < \min\{(l-2) - n/p, (n-2) - n/p\}$ which means that $v = O(|x|^\gamma)$ for all $\gamma > \max\{2-l, 2-n\}$.

Now we have v continuous and bounded on R^n and $\Delta v = \Delta u = -Ku^\sigma$. Therefore $u - v$ is a bounded harmonic function on R^n so $u - v = u_\infty$, a constant. We get $u - u_\infty = v$ so $u - u_\infty = O(|x|^\gamma)$ for all $\gamma > \max\{2-l, 2-n\}$.

We get more terms in the asymptotic expansion of u , with the number of terms depending on how large l is. If $l > n$, we look for a fundamental solution term; that is, a term of the form $f_0/|x|^{n-2}$ where f_0 is a constant.

If $l > n$, then we have $u - u_\infty = O(|x|^\gamma)$ for all $\gamma > 2 - n$. We wish to determine the constant f_0 so that

$$u \sim u_\infty + \frac{f_0}{|x|^{n-2}} \quad \text{as } |x| \rightarrow \infty.$$

Let u_0 be a $C^\infty(R^n)$ function satisfying

$$u_0(x) = \frac{f_0}{|x|^{n-2}} \quad \text{for } |x| \geq 1$$

where f_0 is a constant we wish to determine. We want to have $u - u_\infty - u_0 = O(|x|^\gamma)$ as $|x| \rightarrow \infty$ for all $\gamma > \max\{2-l, 1-n\}$. That implies $u - u_\infty - u_0 \in W_{2,\delta}^p$ for all $\delta < \min\{(l-2) - n/p, (n-1) - n/p\}$. This means we want to have $\Delta(u - u_\infty - u_0) \in \text{Range } \Delta: W_{2,\delta}^p \rightarrow L_{\delta+2}^p$. By (7), this will be true if

$$\int_{R^n} \Delta(u - u_\infty - u_0) dx = 0.$$

By the Divergence Theorem and the fact that u_0 is harmonic for $|x| > 1$,

$$\int_{R^n} \Delta u_0 dx = \int_{|x| \leq 1} \Delta u_0 dx = \int_{|x|=1} \frac{\partial u_0}{\partial n} ds = (2-n)\omega_n f_0.$$

Therefore f_0 must satisfy

$$f_0 = \frac{-1}{(n-2)\omega_n} \int_{R^n} \Delta u dx = \frac{1}{(n-2)\omega_n} \int_{R^n} K u^\sigma dx.$$

This gives us

$$u - u_\infty - \frac{f_0}{|x|^{n-2}} = O(|x|^\gamma) \quad \text{as } |x| \rightarrow \infty \quad \text{for all } \gamma > \max\{2-l, 1-n\}.$$

The process for obtaining more terms in the asymptotic expansion is essentially the same. We look for terms that are Kelvin transforms of homogeneous harmonic polynomials. Suppose $l > n+m$. Assume we have $u_\infty, u_0, u_1, \dots, u_{m-1}$ so that u_∞ is constant, $u_i \in C^\infty(R^n)$,

$$u_i(x) = \frac{f_i}{|x|^{n-2+2i}} \quad \text{for } |x| \geq 1$$

where f_i is a homogeneous harmonic polynomial of degree i and

$$u - u_\infty - u_0 - \dots - u_{m-1} = O(|x|^\gamma) \quad \text{as } |x| \rightarrow \infty \quad \text{for all } \gamma > 2 - (n+m).$$

Let $\mathcal{H}_m$ be the vector space of homogeneous harmonic polynomials of degree m . Choose a basis $\{H_1, \dots, H_M\}$ for $\mathcal{H}_m$ such that $\int_{|x|=1} H_i H_j ds = \delta_{ij}$. We want to find coefficients $a_1, \dots, a_M$ so that the next term in the asymptotic expansion of u is u_m with

$$u_m = \frac{\sum a_i H_i}{|x|^{n-2+2m}} \quad \text{for } |x| \geq 1.$$

To determine the a_i we use the fact that we want

$$\Delta(u - u_\infty - u_0 - \dots - u_m) \in \mathcal{R} = \text{Range } \Delta: W_{2,\delta}^p \rightarrow L_{\delta+2}^p$$

for all δ satisfying

$$(n-2 - \frac{n}{p}) + m < \delta < \min\{(l-2 - \frac{n}{p}), (n-2 - \frac{n}{p}) + m + 1\}.$$

We have

$$\mathcal{R} = \left\{ g \in L_{\delta+2}^p: \int_{R^n} g(x) H(x) dx = 0 \quad \text{for all } H \in \bigcup_{j=0}^m \mathcal{H}_j \right\}$$

so we need

$$\int_{R^n} \Delta(u - u_\infty - u_0 - \cdots - u_m)H(x) dx = 0 \quad \text{for all } H \in \bigcup_{j=0}^m \mathcal{H}_j.$$

Since for $H \in \mathcal{H}_m$

$$\int_{R^n} \Delta(c - u_0 - \cdots - u_{m-1})H dx = 0,$$

it is sufficient to have

$$\int_{R^n} \Delta(u - u_m)H_j(x) dx = 0 \quad \text{for } j = 1, \dots, M.$$

This gives us M conditions which determine $a_1, \dots, a_M$. A computation shows that

$$\int_{R^n} \Delta u H_i(x) dx = \int_{R^n} \Delta u_m H_i(x) dx,$$

if

$$\begin{aligned} a_i &= \frac{1}{2-n-2m} \int_{R^n} \Delta u H_i dx \\ &= \frac{1}{2-n-2m} \int_{R^n} -Ku^\sigma H_i dx. \end{aligned}$$

This completes the proof of Theorem 1.

4. PROOF OF THEOREM 2

Fix $c > 0$. Choose $p > n$ and δ satisfying $-n/p < \delta < \min\{l, n\} - 2 - n/p$. For each $v \in W_{1,\delta}^p$, define

$$f_v = \begin{cases} -K(v+c)^\sigma & \text{if } v+c \geq 0; \\ 0 & \text{if } v+c < 0. \end{cases}$$

Then $f_v \in L_{\delta'+2}^p$ for $\delta' < (l-2) - n/p$ due to the decay of K . Therefore, for any δ' satisfying $\delta < \delta' < \min\{l, n\} - 2 - n/p$, we have

- (i) $f_v \in L_{\delta'+2}^p$ for all $v \in W_{1,\delta}^p$
- (ii) $\Delta: W_{2,\delta'}^p \rightarrow L_{\delta'+2}^p$ is an isomorphism by (5).

Therefore there is a unique $w \in W_{2,\delta'}^p$ with $\Delta w = f_v$. This yields a map

$$T: W_{1,\delta}^p \rightarrow W_{2,\delta'}^p$$

given by $Tv = w$ where $\Delta w = f_v$. Since $W_{2,\delta'}^p \subset W_{1,\delta}^p$ and the inclusion is compact, T is a compact mapping

$$T: W_{1,\delta}^p \rightarrow W_{1,\delta}^p.$$

Suppose $v = \alpha Tv$ for some $\alpha \in [0, 1]$. This means that

$$\Delta v = \begin{cases} -\alpha K(v+c)^\sigma & \text{if } v+c \geq 0; \\ 0 & \text{if } v+c < 0. \end{cases}$$

Since $K \leq 0$, $\Delta v \geq 0$ and since $v \in W_{1,\delta}^p$, $v \rightarrow 0$ as $|x| \rightarrow \infty$ by (4). By the maximum principle, $v \leq 0$ and $v + c \leq c$. Therefore

$$\begin{aligned} \|v\|_{p,1,\delta} &\leq \|v\|_{p,2,\delta} \leq A\|\Delta v\|_{L_{\delta+2}^p} \\ &\leq A\|f_v\|_{L_{\delta+2}^p} \\ &\leq Ac^\sigma \|K\|_{L_{\delta+2}^p}. \end{aligned}$$

The Leray-Schauder Fixed Point Theorem applies with $M = Ac^\sigma \|K\|_{L_{\delta+2}^p}$ and T has a fixed point; that is, there is some $v \in W_{1,\delta}^p$ satisfying $\Delta v = f_v$. K is locally Hölder continuous by hypothesis and v is Hölder continuous because $v \in W_{1,\delta}^p$ and $p > n$. Therefore f_v is Hölder continuous. Since $\Delta v = f_v$, elliptic regularity implies that $v \in C^2$. We shall show that $u = v + c$ is a solution to $\Delta u + Ku^\sigma = 0$. This is true if $v + c \geq 0$ for all x , for then $f_v = -K(v + c)^\sigma$ everywhere.

To show $v + c \geq 0$, assume $v + c < 0$ somewhere, and show this leads to a contradiction. Suppose v has a minimum at x_0 with $v(x_0) + c = a < 0$. Since v is continuous, there is some neighborhood N of x_0 such that $v(x) + c < 0$ for all $x \in N$. Then for all $x \in N$, $\Delta v = 0$. Since v attains its minimum at x_0 , which is in the interior of N , v must be constant on N . Therefore $v + c = a$ for all $x \in N$. Since v is continuous, $v + c = a$ on ∂N . Continuing this, we would get that v is constant on all of R^n . Since $c > 0$ and $a < 0$, v would have to be some negative constant on R^n . However, we know that $v \rightarrow 0$ as $|x| \rightarrow \infty$ because $v \in W_{1,\delta}^p$. Therefore, $v + c \geq 0$ everywhere and $u = v + c$ is a solution to the equation. Since both u and K are bounded and $\sigma > 1$, we have u satisfying $\Delta u - Au \leq 0$ for A sufficiently large. By the Hopf maximum principle, u cannot achieve a nonpositive minimum since we know u is not identically zero. Therefore we have $u > 0$ on R^n .

To complete the proof of Theorem 2, we must show uniqueness. Suppose u_1 and u_2 both satisfy $\Delta u + Ku^\sigma = 0$ and $u_1, u_2 \rightarrow c$ as $|x| \rightarrow \infty$. Then $u_1 - u_2 \rightarrow 0$ as $|x| \rightarrow \infty$. We shall show that $u - v$ cannot attain a positive maximum nor a negative minimum. Suppose $u_1 - u_2$ has a positive maximum at x_0 . Then in a neighborhood of x_0 , $u_1 - u_2 > 0$. Let $\mathcal{U} = \{x: (u_1 - u_2)(x) > 0\}$. On the boundary of $\mathcal{U}$, $u_1 - u_2$ must equal 0. In $\mathcal{U}$, $\Delta(u_1 - u_2) = -K(u_1^\sigma - u_2^\sigma)$ is nonnegative, so $u_1 - u_2$ is subharmonic. Therefore, by the maximum principle, $u_1 - u_2$ must be constant in $\mathcal{U}$. This would mean that $\mathcal{U}$ must be all of R^n . However, we know that $u_1 - u_2 \rightarrow 0$ as $|x| \rightarrow \infty$ and so cannot be a nonzero constant everywhere. Therefore $u_1 - u_2$ cannot have a positive maximum. Similarly, $u_1 - u_2$ cannot have a negative minimum. Since $u_1 - u_2 \rightarrow 0$ as $|x| \rightarrow \infty$, we must have $u_1 = u_2$.

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