PASTING AND MODULAR LATTICES

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ABSTRACT. A classical lattice construction of R. P. Dilworth is the *gluing* of two lattices. A number of recent papers by A. Slavík, A. Day, and J. Ježek investigated a generalization: *pasting*. In this note we prove that by pasting two finite modular lattices, one obtains a modular lattice.

A classical problem of lattice theory (see, e.g., [3, Problems V.10 and V.11]) has recently been solved: Only three lattice varieties have the Amalgamation Property, namely, $\mathbf{T}$, the trivial variety, $\mathbf{D}$, the variety of distributive lattices, and $\mathbf{L}$, the variety of all lattices. This result was proved in two steps. First, G. Grätzer, B. Jónsson, and H. Lakser [4] proved that $\mathbf{T}$ and $\mathbf{D}$ are the only modular lattice varieties that have the Amalgamation Property. Then A. Day and J. Ježek [1] proved that $\mathbf{L}$ is the only nonmodular lattice variety with the Amalgamation Property.

The method of G. Grätzer, B. Jónsson, and H. Lakser [4] is projective geometric: in a modular lattice variety $\mathbf{V}$ having the Amalgamation Property, a lattice L is amalgamated many times with M_3 , the five element modular, nondistributive lattice, obtaining a complemented modular lattice. Then Frink's Theorem [2] gives an embedding of L into a projective geometry.

The method of A. Day and J. Ježek [1] uses a version of amalgamation discussed in A. Slavík [8] (see also G. Grätzer [4, Exercise 12 of Section V.4]); in this paper it will be called *pasting* (see below for the definition). Pasting is stronger than gluing but much weaker than amalgamation. They prove the following theorem: Let $\mathbf{V}$ be a nonmodular variety of lattices; if $\mathbf{V}$ is closed under the pasting of finite lattices, then $\mathbf{V} = \mathbf{L}$.

A. Day and J. Ježek raise the following problem: Can one prove the result of G. Grätzer, B. Jónsson, and H. Lakser on the nonexistence of modular nondistributive lattice varieties having the amalgamation property using this method. More specifically: can one start with M_3 , and obtain N_5 (the five element

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nonmodular lattice) by pasting and by forming sublattices and homomorphic images?

In this note we answer this question in the negative. Namely, we prove the following

Theorem. *The variety $\mathbf{M}$ of all modular lattices is closed under the pasting of finite lattices.*

First, the definition of pasting: Let L be a lattice. Let A , B , and S be sublattices of L , $A \cap B = S$, $A \cup B = L$. Then L pastes A and B together over S , if every amalgamation of A and B over S contains L as a sublattice. More precisely:

Definition. Let L be a lattice. Let A , B , and S be sublattices of L , $A \cap B = S$, $A \cup B = L$. Let f_A and f_B be the embeddings of A and B , respectively, into L . L pastes A and B together over S if whenever g_A and g_B are embeddings of A and B into a lattice K satisfying $xg_A = xg_B$ for all $x \in S$, then there is an embedding h of L into K satisfying $f_A h = g_A$ and $f_B h = g_B$ (see Figure 1).

FIGURE 1

A. Day and J. Ježek [1] sharpen a result of A. Slavík [8] with the following characterization of pasting in the finite case.

Proposition. Let L be a finite lattice. Let A , B , and S be sublattices of L , $A \cap B = S$, $A \cup B = L$. L pastes A and B together over S if and only if the following two conditions hold:

- (1) For $a \in A$ and $b \in B$, if $a < b$, then there exists an $s \in S$ satisfying $a \leq s \leq b$; and dually.
- (2) For $s \in S$, all the covers of s in L are in A or all are in B ; and dually.

This Proposition has the following:

Corollary. Let L be a finite lattice that pastes A and B over S . Let $u, v \in L$ be chosen in such a way that $S \cap [u, v]$ is not empty. Define $L_1 = [u, v]$, $A_1 = A \cap [u, v]$, $B_1 = B \cap [u, v]$, $S_1 = S \cap [u, v]$. Then L_1 pastes A_1 and B_1 over S_1 .

Proof. It is obvious that if L , A , B , and S satisfy (1) and (2), then so do L_1 , A_1 , B_1 , and S_1 .

For elements a, b, c of the lattice L , we shall use the notation $N_5(a, b, c)$ to indicate that a, b, c generate a sublattice isomorphic to N_5 ; more precisely, $N_5(a, b, c)$ means that $a < b$, both a and b are incomparable with c , $a \wedge c = b \wedge c$, and $a \vee c = b \vee c$.

Now we start proving the Theorem. By way of contradiction, let L be a finite nonmodular lattice, and let L be the pasting of the modular sublattices A and B over S . We can choose L as a minimal such lattice.

Since L is nonmodular, it contains elements a, b, c satisfying the condition $N_5(a, b, c)$. Clearly, $a \wedge c = 0$ and $b \vee c = 1$; indeed if $a \wedge c = u$ and $b \vee c = v$, and $0 < u$ or $v < 1$, then applying the Corollary to u and v we get a smaller lattice, contradicting the minimality of L . Thus L is *almost modular*: every proper interval of L is modular.

We can further assume that in L a is the smallest element x satisfying $N_5(x, b, c)$, that is, if $x < a$, then $N_5(x, b, c)$ fails to hold. Dually, we can assume that b is a maximal element for which $N_5(a, b, c)$ holds.

Claim. $a \succ 0$ and $1 \succ b$.

Proof. Let $0 < x < a$ in L . First, we verify that $x = b \wedge (x \vee c)$. Indeed, if $x < b \wedge (x \vee c)$, then $N_5(x, b \wedge (x \vee c), c)$. Now $x \vee c < 1$, since $x \vee c = 1$ would contradict the minimality of a . Thus $N_5(x, b \wedge (x \vee c), c)$ gives us a proper sublattice isomorphic to N_5 , contradicting the minimality of L . This shows that $x = b \wedge (x \vee c)$. This however implies that $N_5(a, b, x \vee c)$, yielding an N_5 with x as the zero, contradicting the minimality of L . This proves the first part of the claim; the second part follows by duality.

So we are given a, b, c satisfying $N_5(a, b, c)$, $a \succ 0$ and $1 \succ b$. We choose u in $[0, c]$ so that $u \succ 0$; we choose v in $[c, 1]$ so that $1 \succ v$. Now we distinguish two cases.

Case 1. $a \vee u = 1$ and $b \wedge v = 0$.

Case 2. $a \vee u < 1$ (or dually).

If Case 1 holds (see Figure 2), then by condition (2) of the Proposition, $a, u \in A$ or B and $b, v \in A$ or B . We cannot have $a, u \in A$ and $b, v \in A$, because then $N_5(a, b, u)$ in A , contradicting the modularity of A . Similarly for B . Hence, by symmetry of A and B , we can assume that $a, u \in A$ and $b, v \in B$. This implies that $0, 1 \in S$.

By condition (1) of the Proposition, we can choose $t \in [a, b]$ and $w \in [u, v]$, $t, w \in S$. Observe that $t \wedge w = 0$ and $t \vee w = 1$. Since $a < t$ or $t < b$, by

FIGURE 2

reason of symmetry we can assume that $a < t$. Now $N_5(a, t, w)$ and all three elements are in A , contradicting the modularity of A .

If Case 2 holds (see Figure 3), then $a \vee u$ covers both a and u since $[0, a \vee u]$ is modular. Since $u < c$, we can choose x in $[u, c]$, $x > u$. We now verify that $N_5(a, b \wedge (a \vee x), x)$. Observe that $[u, c]$ and $[a \vee u, 1]$ are perspective intervals ($a \vee u \vee c = 1$ is obvious and $(a \vee u) \wedge c = u$ follows from the fact that $a \vee u > u$), and $[a \vee u, 1]$ and $[a, b]$ are perspective intervals (use that $1 > b$), so $a \vee x > a \vee u$, and $b \wedge (a \vee x) > a$ by the modularity of $[u, 1]$ and $[a, 1]$. In particular, $a < b \wedge (a \vee x)$. It is obvious that $x \wedge a = x \wedge (b \wedge (a \vee x))$ and $x \vee a = x \vee (b \wedge (a \vee x))$, verifying that $N_5(a, b \wedge (a \vee x), x)$.

$N_5(a, b \wedge (a \vee x), x)$ means that L has a sublattice isomorphic to N_5 in $[0, a \vee x]$. By the minimality of L , we must then have $a \vee x = 1$. Since $a \vee u > u$, $1 > x$, therefore $x = c$. So we get the lattice of Figure 3 as a cover preserving sublattice of L .

Now apply the dual of condition (2) to the element 1; we obtain that b , c , $a \vee u$ are in A or B , say in A . Since these three elements generate the whole lattice, we get that A is nonmodular, a contradiction. This contradiction proves the Theorem.

In conclusion, we mention a few problems.

Problem 1. Does the Theorem hold for infinite lattices?

Problem 2. In [5], continuumly many modular lattice varieties are given that are closed under gluing. Are they also closed under finite pasting?

FIGURE 3

Problem 3. *There are two “extremal” cases of pasting: gluing and the insertion of a lattice in the critical interval of N_5 . Can all pastings be “put together” from these two special cases?*

If a lattice variety $\mathbf{V}$ is closed under gluing, then by G. Grätzer and D. Kelly [5], $\mathbf{V} \circ \mathbf{D}$ (the *variety product* of $\mathbf{V}$ and $\mathbf{D}$) is a variety again. By a result of R. McKenzie and T. Harrison (see T. Harrison [6]), if $\mathbf{V}$ is a nonmodular lattice variety and $\mathbf{V} \circ \mathbf{D}$ is a variety, then $\mathbf{V} = \mathbf{L}$. These two results combine: If $\mathbf{V}$ is a nonmodular variety closed under gluing, then $\mathbf{V} = \mathbf{L}$. This is a sharpening of the Day-Ježek result. The question is whether the finitary version is true.

Problem 4. *If $\mathbf{V}$ is a nonmodular lattice variety and $\mathbf{V}$ is closed under the gluing of finite lattices, is $\mathbf{V} = \mathbf{L}$?*

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