POLYNOMIALS OF GENERATORS OF INTEGRATED SEMIGROUPS

RALPH DELAUBENFELS

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ABSTRACT. We give general sufficient conditions on p and A , for $p(A)$ to generate an exponentially bounded holomorphic k -times integrated semigroup, where p is a polynomial and A is a linear operator on a Banach space. Corollaries include the following.

(1) If iA generates a strongly continuous group and p is a polynomial of even degree with positive leading coefficient, then $-p(A)$ generates a strongly continuous holomorphic semigroup of angle $\frac{\pi}{2}$. (2) If $-A$ generates a strongly continuous holomorphic semigroup of angle θ and p is an n th degree polynomial with positive leading coefficient, with $n(\frac{\pi}{2} - \theta) < \frac{\pi}{2}$, then $-p(A)$ generates a strongly continuous holomorphic semigroup of angle $\frac{\pi}{2} - n(\frac{\pi}{2} - \theta)$. (3) If $(-A)$ generates an exponentially bounded holomorphic k -times integrated semigroup of angle θ , and p and θ are as in (2), then $-p(A)$ generates an exponentially bounded holomorphic $(k+1)$ -times integrated semigroup of angle $\frac{\pi}{2} - n(\frac{\pi}{2} - \theta)$.

1. INTRODUCTION

In this paper, we consider linear operators, A , on a Banach space, whose resolvents $(w - A)^{-1}$, are $O(w^k)$ outside some set K , and polynomials, p , such that $p(K)$ is contained in a sector, and show that $p(A)$ generates a holomorphic $(k+2)$ -times integrated semigroup. When $p(A)$ is densely defined, it generates a $(k+1)$ -times integrated semigroup (Theorem 4). This is applied to polynomials of generators of holomorphic k -times integrated semigroups (Theorem 9) and polynomials of generators of strongly continuous groups (Theorem 11), as described in the abstract. Even for $k = 0$, that is, for polynomials of generators of strongly continuous semigroups, these results are new, except for some special cases, which we will describe below.

If k is a natural number, then the strongly continuous family of bounded operators $\{S(t)\}_{t \geq 0}$, is an *exponentially bounded k -times integrated semigroup generated by A* if $S(0) = 0$, and there exists real w such that (w, ∞) is

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contained in the resolvent set of A , with

$$(r - A)^{-1} = r^k \int_0^\infty e^{-rt} S(t) dt, \quad \text{for } r > w$$

(see Arendt [1], [2], Hieber and Kellermann [8], Neubrander [10], Thieme [15]). For convenience, we will say *0-times integrated semigroup* to mean a strongly continuous semigroup. When A has nonempty resolvent set, A generates an exponentially bounded k -times integrated semigroup if and only if the abstract Cauchy problem $u'(t) = A(u(t))$, $u(0) = x$ has a unique exponentially bounded solution, for all x in $D(A^{k+1})$ (see Neubrander [10], or [5]).

It is well-known that the square of a generator of a strongly continuous group generates a C_0 holomorphic semigroup of angle $\frac{\pi}{2}$. (This is the special case of Theorem 11, when $p(t) = t^2$. See Goldstein [7, Chapter 2, 8.7].) The bounded holomorphic semigroup analogue of Theorem 9, for the special case of $p(t) = t^n$, is in [3], where a different method of proof was used. (See Corollary 5 of this paper.) The special case of Theorem 9, when $p(t) = t^2$, appears in Goldstein [6]. In that paper, it is also shown that, if $-A$ generates a cosine function, then $-A^{2n}$ generates a C_0 holomorphic semigroup of angle $\frac{\pi}{2}$, for $n = 1, 2, \dots$.

All operators are linear, on a Banach space, X . We will write e^{tA} for the semigroup generated by A , $D(A)$ for the domain of A . Basic material on strongly continuous semigroups may be found in Goldstein [7], Pazy [14], or van Casteren [16]. When p is a polynomial, $p(A)$ is defined in the obvious way: if $p(t) = \sum_{k=0}^n a_k t^k$, then $p(A) \equiv \sum_{k=0}^n a_k A^k$, with $D(p(A)) = D(A^n)$. All polynomials are complex valued.

2. MAIN RESULTS

Definition 1. $S_\theta = \{re^{i\phi} | r \geq 0, |\phi| < \theta\}$.

Definition 2. A C_0 (strongly continuous) semigroup is a C_0 holomorphic semigroup of angle θ ($0 < \theta \leq \frac{\pi}{2}$) if it extends to a semigroup holomorphic in the interior of S_θ , and continuous on $\overline{S}_\psi$, whenever $\psi < \theta$.

A C_0 holomorphic semigroup of angle θ is a *bounded holomorphic semigroup (BHS)* of angle θ if it is bounded on S_ψ , whenever $\psi < \theta$.

Remark. It is well-known (see any of the references for C_0 semigroups) that $-A$ generates a BHS of angle θ if and only if $D(A)$ is dense, the spectrum of A is contained in $\overline{S}_{\pi/2-\theta}$ and for all $\psi > (\frac{\pi}{2} - \theta)$, $\{\|w(w - A)^{-1}\| | w \notin \overline{S}_\psi\}$ is bounded.

Definition 3. Suppose $\frac{\pi}{2} \geq \theta > 0$. The k -times integrated semigroup $S(t)$ is an *exponentially bounded holomorphic k -times integrated semigroup of angle θ* if it extends to a family of operators $\{S(z)\}_{z \in S_\theta}$ satisfying

- (1) The map $z \rightarrow S(z)$, from S_θ into $B(X)$, is holomorphic.
- (2) $d^k/dz^k S(z)$ is a semigroup.

- (3) For all $\psi < \theta$, $S(z)$ is strongly continuous on $\overline{S_\psi}$.
- (4) For all $\psi < \theta$, there exist finite M_ψ , w_ψ such that $\|S(z)\| \leq M_\psi e^{w_\psi|z|}$, for all z in S_ψ .

Remark. Essentially the same idea, as appears in Definition 3, when $D(A)$ is dense, is in Okazawa [13], where semigroups of class (H_n) are defined.

Remark. In [4], we show that, if $D(A)$ is dense, then $-A$ generates an exponentially bounded holomorphic k -times integrated semigroup of angle $\theta > 0$ if and only if for all $\psi > (\frac{\pi}{2} - \theta)$, there exists real c_ψ such that the spectrum of A is contained in $c_\psi + S_\psi$, with $\{\|w^{1-k}(w-A)^{-1}\| | w \notin c_\psi + S_\psi\}$ bounded. When $D(A)$ is not dense, this condition is sufficient to guarantee that $-A$ generates an exponentially bounded holomorphic $(k+1)$ -times integrated semigroup of angle θ .

Theorem 4. Suppose K is a subset of the complex plane containing the spectrum of B , k is a nonnegative integer, $\frac{\pi}{2} > \theta > 0$, $\{\|w^{1-k}(w-A)^{-1}\| | w \notin K\}$ is bounded, and q is a polynomial such that $q(K)$ is contained in S_θ . Then

- (a) $-q(A)$ generates an exponentially bounded holomorphic $(k+1)$ -times integrated semigroup of angle $(\frac{\pi}{2} - \theta)$.
- (b) If $D(q(A))$ is dense, then $-q(A)$ generates an exponentially bounded holomorphic k -times integrated semigroup of angle $(\frac{\pi}{2} - \theta)$.
- (c) If $k = 0$, $D(q(A))$ is dense, and $q(0) = 0$, then $-q(A)$ generates a BHS of angle $(\frac{\pi}{2} - \theta)$.

Proof. There exists finite M such that

$$\|(w-A)^{-1}\| \leq M|w|^{k-1}, \quad \forall w \notin K.$$

Let $p(t) = q(t) - q(0)$, $V = S_\theta - q(0)$. To prove the theorem, it is sufficient to show that the spectrum of $p(A)$ is contained in V , with $\{\|z^{1-k}(z-p(A))^{-1}\| | z \notin V\}$ bounded (see Remarks after Definitions 2 and 3).

Suppose z is not in V . Let $\{w_j\}_{j=1}^N$ be the (not necessarily distinct) zeroes of $z - p(w)$, that is,

$$z - p(w) = \prod_{j=1}^N (w_j - w), \quad \forall \text{ complex } w.$$

We have

$$z - p(A) = \prod_{j=1}^N (w_j - A).$$

For any j , since $p(w_j) = z$ is not in V , w_j is not in K . Thus $(w_j - A)$ is invertible, and $\|(w_j - A)^{-1}\| \leq M|w_j|^{k-1}$. Thus, $z - p(A)$ is invertible, and

we obtain the following upper bound for $(z - p(A))^{-1}$.

$$\begin{aligned} \|(z - p(A))^{-1}\| &\leq \prod_{j=1}^N \|(w_j - A)^{-1}\| \\ &\leq M^N \prod_{j=1}^N |w_j|^{k-1} \\ &= M^N |z|^{k-1}. \end{aligned}$$

Thus, $\{\|z^{1-k}(z - p(A))^{-1}\| | z \notin V\}$ is bounded, proving the theorem. $\square$

As an immediate corollary, we get the results of [3].

Corollary 5. Suppose $-A$ generates a BHS of angle θ , and $n(\frac{\pi}{2} - \theta) < \frac{\pi}{2}$. Then $-A^n$ generates a BHS of angle $\frac{\pi}{2} - n(\frac{\pi}{2} - \theta)$.

Proof. Suppose $\frac{\pi}{2} > \psi > n(\frac{\pi}{2} - \theta)$. Then, since $-A$ generates a BHS of angle θ , the spectrum of A is contained in $S_{\psi/n}$, and $\{\|z(z - A)^{-1}\| | z \notin S_{\psi/n}\}$ is bounded.

Let $q(t) = t^n$. Since A generates a BHS, $D(q(A))$ is dense. Also $q(S_{\psi/n})$ is contained in S_ψ , so by Theorem 4(c), $-A^n = -q(A)$ generates a BHS of angle $\frac{\pi}{2} - \psi$, whenever $\frac{\pi}{2} > \psi > n(\frac{\pi}{2} - \theta)$. This implies that $-A^n$ generates a BHS of angle $\frac{\pi}{2} - n(\frac{\pi}{2} - \theta)$. $\square$

In order to apply Theorem 4 to more general polynomials of other generators, we need some elementary lemmas.

Lemma 6. Suppose E is a subset of the complex plane, and $\theta \geq 0$. Then

$$\overline{\lim}_{R \rightarrow \infty} \sup\{|\arg(z)| | z \in E, |z| = R\} \leq \theta$$

if and only if for all $\psi > \theta$, there exists real c_ψ such that E is contained in $c_\psi + S_\psi$.

Proof. Suppose the $\overline{\lim}$ inequality holds, and $\psi > \theta$. There exists finite M such that $|\arg(z)| < \psi$, when z is in E and $|z| \geq M$. Thus, E is contained in $S_\psi \cup \{z \in \mathbb{C} | |z| \leq M\}$, which may be shown to be contained in $-M(1 + \cot \psi) + S_\psi$.

Conversely, suppose that, for all $\psi > \theta$, there exists real c_ψ such that E is contained in $c_\psi + S_\psi$. For any $\psi \leq \pi$, it is not hard to see that $\overline{\lim}_{R \rightarrow \infty} \sup\{|\arg(z)| | z \in c_\psi + S_\psi, |z| = R\}$ equals ψ . Thus

$$\overline{\lim}_{R \rightarrow \infty} \sup\{|\arg(z)| | z \in E, |z| = R\} \leq \psi,$$

for all $\psi > \theta$, which concludes the proof. $\square$

Lemma 7. If $p(t) = t^n + q(t)$, where q is a polynomial of degree less than n , $\theta \geq 0$ and c is real, then, for all $\psi > n\theta$, there exists real c_ψ such that $p(c + S_\theta)$ is contained in $c_\psi + S_\psi$.

Proof. $\lim_{|z| \rightarrow \infty} p(c+z)/z^n = 1$. Thus,

$$\begin{aligned} & \overline{\lim}_{R \rightarrow \infty} \sup\{|\arg(z)| \mid z \in p(c+S_\theta), |z| = R\} \\ &= \overline{\lim}_{R \rightarrow \infty} \sup\{|\arg(p(c+z))| \mid z \in S_\theta, |z| = R\} \\ &= \overline{\lim}_{R \rightarrow \infty} \sup\{|\arg(z^n)| \mid z \in S_\theta, |z| = R\} \\ &= n\theta. \end{aligned}$$

Applying Lemma 6 now gives the result. $\square$

Lemma 8. Suppose K equals $-c+S_\theta \cup c-S_\theta$, where c and θ are nonnegative, and $p(t) = t^{2n} + q(t)$, where q is a polynomial of degree less than $2n$. Then, for all $\psi > 2n\theta$, there exists real c_ψ such that $p(K)$ is contained in $c_\psi + S_\psi$.

Proof. Let $K^+ = -c + S_\theta$. Since $K = K^+ \cup -K^+$, it is sufficient, by Lemma 6, to show that $\overline{\lim}_{r \rightarrow \infty} \sup\{|\arg(p(z))| \mid z \in \pm K^+\} = 2n$; this follows exactly as in the proof of Lemma 7. $\square$

Theorem 9. Suppose $-A$ generates an exponentially bounded holomorphic k -times integrated semigroup of angle θ , $p(t) = t^n + q(t)$, where q is a polynomial of degree less than n , and $n(\frac{\pi}{2} - \theta) < \frac{\pi}{2}$. Then

- (a) $-p(A)$ generates an exponentially bounded holomorphic $(k+1)$ -times integrated semigroup of angle $\frac{\pi}{2} - n(\frac{\pi}{2} - \theta)$.
- (b) If $D(p(A))$ is dense, then $-p(A)$ generates an exponentially bounded holomorphic k -times integrated semigroup of angle $\frac{\pi}{2} - n(\frac{\pi}{2} - \theta)$.
- (c) If $k = 0$, then $-p(A)$ generates a C_0 holomorphic semigroup of angle $\frac{\pi}{2} - n(\frac{\pi}{2} - \theta)$.

Proof. Suppose $\frac{\pi}{2} > \psi > n(\frac{\pi}{2} - \theta)$. Choose ϕ such that $\frac{\pi}{n} > \phi > \frac{\pi}{2} - \theta$. Since $-A$ generates an exponentially bounded holomorphic k -times integrated semigroup of angle θ , there exists real c such that the spectrum of A is contained in $c + S_\phi$, and $\{\|w^{1-k}(w-A)^{-1}\| \mid w \notin c + S_\phi\}$ is bounded.

By Lemma 7, there exists real c_ψ such that $p(c+S_\phi)$ is contained in $c_\psi + S_\psi$. By Theorem 4(a), $c_\psi I - p(A)$ generates an exponentially bounded holomorphic $(k+1)$ -times integrated semigroup of angle $\frac{\pi}{2} - \psi$.

Thus, whenever $\frac{\pi}{2} > \psi > n(\frac{\pi}{2} - \theta)$, $-p(A)$ generates an exponentially bounded holomorphic $(k+1)$ -times integrated semigroup of angle $\frac{\pi}{2} - \psi$. This implies (a).

The same argument, using Theorem 4(b), implies (b).

For (c), note that, since $-A$ generates a C_0 semigroup, $D(p(A))$ is dense. Thus the argument above, with Theorem 4(c), implies (c). $\square$

Corollary 10. Suppose p is a polynomial with positive leading coefficient.

- (a) If $-A$ generates a C_0 holomorphic semigroup of angle $\frac{\pi}{2}$, then $-p(A)$ generates a C_0 holomorphic semigroup of angle $\frac{\pi}{2}$.

- (b) If $-A$ generates an exponentially bounded holomorphic k -times integrated semigroup of angle $\frac{\pi}{2}$, then $-p(A)$ generates an exponentially bounded holomorphic $(k+1)$ -times integrated semigroup of angle $\frac{\pi}{2}$.

The following theorem would follow from Theorem 9 and the fact that the square of the generator of a C_0 group generates a C_0 holomorphic semigroup, if $q(t)$ contained only even powers of t . Theorem 11 is more general, in that q may be any polynomial of degree less than $2n$.

Theorem 11. Suppose iA generates a strongly continuous group, and $p(t) = t^{2n} + q(t)$, where q is a polynomial of degree less than $2n$. Then $-p(A)$ generates a C_0 holomorphic semigroup of angle $\frac{\pi}{2}$.

Proof. Suppose $\frac{\pi}{2} > \phi > 0$. Since iA generates a C_0 group, there exists positive r such that the spectrum of A is contained in the horizontal strip $\{z \in \mathbb{C} \mid |\operatorname{Im}(z)| < r\}$, with $\{\| \operatorname{Im}(z)(z - A)^{-1} \| \mid |\operatorname{Im}(z)| \geq r\}$ bounded.

Let $c = r \cot(\frac{\phi}{4n})$, $K = (-c + S_{\phi/4n}) \cup (c - S_{\phi/4n})$.

Since $\{z \in \mathbb{C} \mid |\operatorname{Im}(z)| < r\}$ is contained in K , and $\{\|z/\operatorname{Im}(z)\| \mid z \notin K\}$ is bounded, it follows that $\{\|z(z - A)^{-1}\| \mid z \notin K\}$ is bounded.

By Lemma 8, there exists real c_ϕ such that $p(K)$ is contained in $c_\phi + S_\phi$.

Since iA generates a C_0 group, $D(p(A))$ is dense. By Theorem 4(c), $c_\phi I - p(A)$ generates a BHS of angle $\frac{\pi}{2} - \phi$. Thus, for any positive ϕ , $-p(A)$ generates a C_0 holomorphic semigroup of angle $\frac{\pi}{2} - \phi$, so that $-p(A)$ generates a C_0 holomorphic semigroup of angle $\frac{\pi}{2}$. $\square$

The same argument, using Theorem 4(a) and (b), instead of (c), gives the following.

Theorem 12. Suppose both iA and $-iA$ generate exponentially bounded k -times integrated semigroups and p is as in Theorem 11. Then

- (a) $-p(A)$ generates an exponentially bounded holomorphic $(k+1)$ -times integrated semigroup of angle $\frac{\pi}{2}$.
 (b) If $D(p(A))$ is dense, then $-p(A)$ generates an exponentially bounded holomorphic k -times integrated semigroup of angle $\frac{\pi}{2}$.

Corollary 13. Suppose p is an arbitrary polynomial.

- (a) If both iA and $-iA$ generate exponentially bounded k -times integrated semigroups, then $-|p|^2(A)$ generates an exponentially bounded holomorphic $(k+1)$ -times integrated semigroup of angle $\frac{\pi}{2}$.

- (b) If, in addition to (a), $D(|p|^2(A))$ is dense, then $-|p|^2(A)$ generates an exponentially bounded holomorphic k -times integrated semigroup of angle $\frac{\pi}{2}$.
- (c) If iA generates a C_0 group, then $-|p|^2(A)$ generates a C_0 holomorphic semigroup of angle $\frac{\pi}{2}$.

3. EXAMPLES

The most obvious application of Theorem 11 is to choose $A = i\frac{d}{dx}$, on $L^p(\mathbf{R})$, for $1 \leq p < \infty$. For $p = \infty$, the resolvent of iA still satisfies the same growth conditions as the generator of a C_0 group, thus the proof of Theorem 11 with Theorem 4(a) replacing Theorem 4(c), gives us (b) of the following.

Example 1. Let $A \equiv i\frac{d}{dx}$, on $L^p(\mathbf{R})$ ($1 \leq p \leq \infty$), with maximal domain, and $B \equiv (-1)^n(\frac{d}{dx})^{2n} + q(i\frac{d}{dx}) = A^{2n} + q(A)$, where q is a polynomial of degree less than $2n$.

(a) If $1 \leq p < \infty$, then B generates a C_0 holomorphic semigroup of angle $\frac{\pi}{2}$.

(b) If $p = \infty$, then B generates an exponentially bounded holomorphic once-integrated semigroup of angle $\frac{\pi}{2}$.

Remark. In Hieber and Kellermann [8], it is shown that $Q(i\frac{d}{dx})$, on $L^p(\mathbf{R})$, $1 \leq p \leq \infty$, generates an exponentially bounded once-integrated semigroup, whenever $Q(\mathbf{R})$ is contained in a left half-plane.

Suppose B generates an exponentially bounded k -times integrated semigroup. In Neubrander [11], and [5], it is shown that

$$(1) \quad A \equiv \begin{bmatrix} B & B \\ 0 & B \end{bmatrix}, \quad D(A) \equiv D(B) \times D(B),$$

generates an exponentially bounded $(k+1)$ -times integrated semigroup. It is straightforward to show that this integrated semigroup is holomorphic if the integrated semigroup generated by B is (see Neubrander and deLaubenfels [12]). Thus we have the following, using the fact that

$$A^n = \begin{bmatrix} B^n & nB^n \\ 0 & B^n \end{bmatrix},$$

for all n .

Example 2. (a) Suppose both iB and $-iB$ generate exponentially bounded k -times integrated semigroups, and $p(t) = t^{2n} + q(t)$, where $\deg(q) < 2n$. Then

$$(2) \quad - \begin{bmatrix} p(B) & Bp'(B) \\ 0 & p(B) \end{bmatrix}$$

generates a holomorphic exponentially bounded $(k+2)$ -times integrated semigroup of angle $\frac{\pi}{2}$. If $D(B^{2n})$ is dense, then it generates a holomorphic exponentially bounded $(k+1)$ -times integrated semigroup.

(b) Suppose $-B$ generates an exponentially bounded holomorphic k -times integrated semigroup of angle θ , and p is an n th degree polynomial with positive leading coefficient, with $n(\frac{\pi}{2} - \theta) < \frac{\pi}{2}$. Then the operator in (3.2) generates an exponentially bounded holomorphic $(k+2)$ -times integrated semigroup of angle $\frac{\pi}{2} - n(\frac{\pi}{2} - \theta)$. If $D(B^n)$ is dense, then it generates an exponentially bounded holomorphic $(k+1)$ -times integrated semigroup of angle $\frac{\pi}{2} - n(\frac{\pi}{2} - \theta)$.

In Example 2(a), B could be $Q(i\frac{d}{dx})$, on $L^p(\mathbf{R})$ ($1 \leq p \leq \infty$), when $Q(\mathbf{R})$ is contained in a left half-plane (see Remark after Example 1).

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DEPARTMENT OF MATHEMATICS, OHIO UNIVERSITY, ATHENS, OHIO 45701