THE ASYMPTOTICS OF THE DETERMINANT FUNCTION FOR A CLASS OF OPERATORS

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ABSTRACT. Let A be an elliptic pseudodifferential operator on a closed manifold M and $\text{ord } A > \dim M$. We derive the asymptotics of $\log \det(1 + \varepsilon A^{-1})$ when $\varepsilon \rightarrow \infty$. The constant term of this asymptotics equals $-\log \det A$.

INTRODUCTION

Let A be an elliptic pseudodifferential operator on a closed manifold M and $n = \text{ord } A > \dim M = d$. Suppose that $\arg(A\varphi, \varphi) \leq \alpha$, $0 \leq \alpha < \pi/2$. One can define complex powers of the A and A^{-s} are operators from the trace class when $\text{Re } s > d/n$. The ζ -function $\zeta(s) = \text{Tr } A^{-s}$ is holomorphic in the half-plane $\text{Re } s > d/n$ and admits analytic continuation to a meromorphic function in the whole complex plane. The poles of this function are located in the points $(d - j)/n$; $j = 0, 1, 2, \dots$. The point 0 is not really a pole of $\zeta(s)$. If the operator A is differential or it is a power of a differential operator some other poles drop. The residues of $\zeta(s)$ and the numbers $\zeta(0)$, $\zeta(-1)$, ... can be calculated if one knows the complete symbol of the operator A [1-3]. Particularly the point $s = 0$ is a regular point of the ζ -function and one can define

$$(1) \quad \mathcal{W}(A) = \log \det A = -\zeta'(0).$$

It is easy to check that this definition gives us the generalization of a finite dimensional operator's determinant.

On the other hand the inverse operator A^{-1} belongs to the trace class and the determinants

$$(2) \quad D(\varepsilon) = \det(1 + \varepsilon A^{-1}) = \prod_{j=1}^{\infty} (1 + \varepsilon \lambda_j) \quad (\lambda_j = \mu_j^{-1})$$

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are defined. The infinite product in the right hand side of (2) converges. It is more convenient to consider the function

$$(3) \quad d(\varepsilon) = \log D(\varepsilon) = \sum_{j=1}^{\infty} \log(1 + \varepsilon \lambda_j).$$

We are going to investigate the asymptotic expansion of the $d(\varepsilon)$ when $\varepsilon \rightarrow +\infty$. In particular the determinant $W(A)$ will appear in this expansion.

We think there are two reasons why this expansion is interesting. The first reason is the connection between the Fredholm determinant $\det(1 + \varepsilon A^{-1})$ and the determinant of the elliptic operator A we shall obtain from this expansion. The second reason is that the value $\det(1 + \varepsilon A^{-1})$ appears in the measure theory as the result of integration of $\exp(-\varepsilon(x, x)/2)$ over a Hilbert space with respect to a Gaussian measure with average 0 and correlation operator A^{-1} (e.g. see [4]).

The fact that $\det(1 + \varepsilon A^{-1})$ admits asymptotic expansion for big ε is not surprising. Let us differentiate formally the equality (3) with respect to ε :

$$d'(\varepsilon) = \sum_{j=1}^{\infty} \lambda_j (1 + \varepsilon \lambda_j)^{-1} = \sum_{j=1}^{\infty} (\mu_j + \varepsilon)^{-1} = \text{Tr}(A + \varepsilon)^{-1}.$$

The trace of the resolvent $\text{Tr}(A + \varepsilon)^{-1}$ admits asymptotic expansion when $\varepsilon \rightarrow \infty$ [5]. So one can justify the possibility of the last expansion's integration and receive the asymptotics for $d(\varepsilon)$. The only thing we can not get in this way is the constant of integration, i.e. the term with ε^0 . We shall see later that this term is of the most interest. So we shall derive the asymptotics for $d(\varepsilon)$ directly.

The aim of this paper is to prove

Theorem. *The function $d(\varepsilon)$ admits asymptotic expansion*

$$(4) \quad d(\varepsilon) \sim \sum_{k=-d}^{\infty} p_k \varepsilon^{-k/n} + \sum_{j=0}^{\infty} q_j \varepsilon^{-j} \log \varepsilon$$

with

$$(5) \quad p_0 = -\log \det A.$$

2. CONNECTION BETWEEN $d(\varepsilon)$ AND $\zeta(\varepsilon)$

In this section we shall prove

Proposition 1. *Let $d/n < a < 1$. Then*

$$(6) \quad d(\varepsilon) = \frac{1}{2\pi i} \int_{\text{Re } s=a} \varepsilon^s b(s) \zeta(s) ds$$

where

$$(7) \quad b(s) = \frac{1}{s} \int_0^{\infty} \frac{t^{-s}}{1+t} dt.$$

Proof. It is convenient to introduce the new argument $\omega = \log \varepsilon$. Let $d^*(\varepsilon) = d(e^\omega)$. Then formula (6) can be rewritten in the form

$$(8) \quad d^*(\omega) = \frac{1}{2\pi i} \int_{\operatorname{Re} s=a} e^{s\omega} b(s) \zeta(s) ds.$$

Integrating by parts and changing the argument $t = e^\omega$ we derive

$$b(s) = \int_0^\infty t^{-s-1} \log(1+t) dt = \int_{-\infty}^\infty e^{-\omega s} \log(1+e^\omega) d\omega.$$

Note that the last formula gives the Fourier transformation $\log(1+e^\omega) \rightarrow b(is)$. So by the inverse Fourier formula

$$\log(1+e^\omega) = \frac{1}{2\pi i} \int_{\operatorname{Re} s=a} b(s) e^{\omega s} ds.$$

After substituting $\varepsilon \lambda$ instead of e^ω we derive

$$\log(1+\varepsilon \lambda) = \frac{1}{2\pi i} \int_{\operatorname{Re} s=a} \varepsilon^s b(s) \lambda^s ds.$$

Now to obtain (6) we have to sum the last identities for $\lambda = \lambda_j$. Note that the summation and integration operations in the right hand side commute and $\zeta(s)$ is the sum of λ_j^s .

3. ANALYTICAL PROPERTIES OF $b(s)$

We shall obtain asymptotic expansion for $d^*(\omega)$ (or $d(\omega)$) by shifting the contour in (8) to the left. So we must have analytical continuation of $b(s)$ into the left half-plane and we must estimate $|b(s)|$ and $|\zeta(s)|$ for large $|\operatorname{Im} s|$. In this section we shall prove

Proposition 2. *The function $b(s)$ admits analytical continuation to a meromorphic function into the half-plane $\operatorname{Re} s < 1$. It has a pole of order 2 into the point $s = 0$ and simple poles into the points $-1, -2, \dots$; $\operatorname{Res}_{s=0} b(s) = 0$, $\operatorname{Res}_{s=0} s b(s) = 1$. The function $b(s)$ satisfies the following estimate*

$$(9) \quad |b(\sigma + i\tau)| \leq C(\sigma) |\tau|^{\sigma-1} \exp\left(-\frac{\pi}{2} |\tau|\right), \quad \sigma < 1, |\tau| \geq 1.$$

Proof. Let

$$J_n = \int_0^\infty t^{-s} (1+t)^n dt, \quad n \geq 1.$$

Integrating by parts we obtain

$$J_n = -\frac{n}{s-1} \int_0^\infty \frac{t^{1-s}}{(1+t)^{n+1}} dt = -\frac{n}{s-1} J_n + \frac{n}{s-1} J_{n+1}.$$

So

$$\frac{s+n-1}{s-1} J_n = \frac{n}{s-1} J_{n+1}$$

and

$$J_n = \frac{n}{s+n-1} J_{n+1}.$$

Thus

$$(10) \quad J_1 = \frac{k!}{s(s+1)\cdots(s+k-1)} \int_0^\infty t^{-s} (1+t)^{-k-1} dt.$$

The integral in the right hand side of (10) is absolutely convergent in the strip $-k < \operatorname{Re} s < 1$. So the assertion about analytical continuation of $b(s)$ is proved.

Applying (10) with $k = 1$ we get

$$\begin{aligned} \operatorname{Res}_{s=0} s b(s) &= \int_0^\infty \frac{dt}{(1+t)^2} = 1; \\ \operatorname{Res}_{s=0} b(s) &= \frac{d}{ds} \int_0^\infty \frac{t^{-s}}{(1+t)^2} dt \Big|_{s=0} = - \int_0^\infty \frac{\log t}{(1+t)^2} dt. \end{aligned}$$

Changing the argument $t \rightarrow 1/u$ in the last integral one can see that

$$\int_0^\infty \frac{\log t}{(1+t)^2} dt = - \int_0^\infty \frac{\log u}{(1+u)^2} du,$$

so

$$\operatorname{Res}_{s=0} b(s) = 0.$$

To prove (9) we shall use the representation (10). The number k in (10) will be large for large $|\tau|$ (really $k \sim |\tau|^2$). Suppose that $-\infty < \sigma_0 < \sigma < \sigma_1 < 1$ with some σ_0 and σ_1 . All constants in the following estimates will depend on σ_0 and σ_1 only. We shall not numerate these constants. They will be designated by the same letter C .

To begin with, let us estimate the integral (10). This integral splits into

$$J = J_1 + J_2 = \int_0^{1/k} t^{-s} (1+t)^{-k-1} dt + \int_{1/k}^\infty t^{-s} (1+t)^{-k-1} dt.$$

The estimation of J_1 is very easy:

$$|J_1| \leq C \int_0^{1/k} t^{-\sigma} dt \leq C k^{\sigma-1}.$$

To estimate J_2 let us suppose that $0 \leq \sigma \leq 1$ (note that we shift s from the strip $\sigma_0 < \sigma < \sigma_1$!). The function $t^{-\sigma}$ decreases, so

$$|J_2| \leq k^\sigma \int_{1/k}^\infty (1+t)^{-(k+1)} dt \leq k^{\sigma-1}.$$

If $\sigma < 0$ we shall integrate by parts $n = \lceil \sigma \rceil$ times ($\lceil x \rceil$ —entire part of the number x).

$$\begin{aligned}
 |J_2| &\leq \int_{1/k}^{\infty} t^{-\sigma} (1+t)^{-k-1} dt \\
 &= k^{\sigma-1} \left(1 + \frac{1}{k}\right)^k - \frac{\sigma}{k} \int_{1/k}^{\infty} t^{-\sigma-1} (1+t)^{-k} dt = \dots \\
 &= k^{\sigma-1} \left(1 + \frac{1}{k}\right)^{-k} - \sigma \frac{k^{\sigma}}{k-1} \left(1 + \frac{1}{k}\right)^{-(k-1)} + \dots \\
 &\quad + (-1)^n \frac{\sigma(\sigma+1) \dots (\sigma+n-1)}{k(k-1) \dots (k-n+1)} \int_{1/k}^{\infty} t^{-fr(\sigma)} (1+t)^{-k+n-1} dt.
 \end{aligned}$$

Every term except the last is obviously estimated by $Ck^{\sigma-1}$. The integral in the last term is the integral of the form we have just investigated ($k \mapsto k-n$; $\sigma \mapsto fr(\sigma)$), so it is estimated by $k^{fr(\sigma)-1} (fr(\sigma) = \sigma - \lceil \sigma \rceil)$. The product which stands before the integral is of the order $k^{-n} = k^{-\lceil \sigma \rceil}$. Thus the last term is also estimated by $k^{\sigma-1}$. Finally,

$$|J_2| \leq Ck^{\sigma-1}$$

and

$$(11) \quad \left| \int_0^{\infty} t^{-s} (1+t)^{-k-1} dt \right| \leq Ck^{\sigma-1}.$$

Now we intend to estimate the product which stands before the integral in the right hand side of (10). Denote

$$r(k) = \log \left| \frac{k!}{s^2(s+1) \dots (s+k-1)} \right|$$

and

$$l = \lceil \sigma_0 \rceil + 1.$$

We have

$$\begin{aligned}
 (12) \quad r(k) &= \log k! - 2 \log |s| - \sum_{j=1}^{l-1} \log |s+j| - \sum_{j=l}^{k-1} \log |s+j| \\
 &\leq \log k! - (l+1) \log |\tau| - \frac{1}{2} \sum_{j=1}^{k-1} \log ((\sigma+j)^2 + \tau^2) \\
 &= \log k! - (l+1) \log |\tau| - \sum_{j=1}^{k-1} \log (\sigma+j) - \frac{1}{2} \sum_{j=l}^{k-1} \log \left(1 + \frac{\tau^2}{(\sigma+j)^2}\right) \\
 &= \log l! + \sum_{j=0}^{k-l-1} \log \left(\frac{k-j}{k-j+\sigma-1}\right) - \frac{1}{2} \sum_{j=l}^{k-1} \log \left(1 + \frac{\tau^2}{(\sigma+j)^2}\right) \\
 &\quad - (l+1) \log |\tau|.
 \end{aligned}$$

The first term in the right hand side of (12) is constant. The second term is bounded by $C + (1 - \sigma) \log k$. Indeed,

$$\begin{aligned}
 \sum_{j=0}^{k-1} \log \left(\frac{k-j}{k-j+\sigma-1} \right) &= \sum_{j=0}^{k-l-1} \log \left(1 + \frac{1-\sigma}{k-j+\sigma-1} \right) \\
 &= \sum_{j=0}^{k-l-1} \log \left(1 + \frac{1-\sigma}{l+\sigma+j} \right) \\
 &\leq \log \left(1 + \frac{1-\sigma}{l+\sigma} \right) + \int_1^{k-l} \log \left(1 + \frac{1-\sigma}{x} \right) dx \\
 &= C + x \log \left(1 + \frac{1-\sigma}{x} \right) \Big|_1^{k-l} \\
 &\quad + (1-\sigma) \int_1^{k-l} \frac{dx}{(x+1-\sigma)} \\
 &\leq C + (1-\sigma) \log k.
 \end{aligned}$$

To estimate the third term in the right hand side of (12) we note that the function $\log(1 + \tau^2/(\sigma + j)^2)$ is decreasing with respect to j . Thus

$$\begin{aligned}
 \frac{1}{2} \sum_{j=l}^{k-1} \log \left(1 + \frac{\tau^2}{(\sigma + j)^2} \right) &\geq \frac{1}{2} \int_l^k \log \left(1 + \frac{\tau^2}{(\sigma + x)^2} \right) dx \\
 &= \frac{1}{2} \int_{l+\sigma}^{k+\sigma} \log \left(1 + \frac{\tau^2}{x^2} \right) dx \\
 &= \frac{1}{2} (k + \sigma) \log \left(1 + \frac{\tau^2}{(\sigma + k)^2} \right) \\
 &\quad - \frac{1}{2} (l + \sigma) \log \left(1 + \frac{\tau^2}{(\sigma + l)^2} \right) \\
 &\quad + \tau \arctan \frac{k + \sigma}{\tau} - \tau \arctan \frac{l + \sigma}{\tau} \\
 &\geq C + \frac{1}{2} (k + \sigma) \log \left(1 + \frac{\tau^2}{(\sigma + k)^2} \right) - (l + \sigma) \log |\tau| \\
 &\quad + |\tau| \arctan \frac{k + \sigma}{|\tau|}.
 \end{aligned}$$

Finally

$$\begin{aligned}
 (13) \quad r(k) &\leq C + (1 + \sigma) \log k - \frac{1}{2}(k + \sigma) \log \left(1 + \frac{\tau^2}{(\sigma + k)^2} \right) \\
 &\quad + (l + \sigma) \log |\tau| - |\tau| \arctan \frac{k + \sigma}{|\tau|} - (l + 1) \log |\tau| \\
 &= C + (1 - \sigma) \log k + (\sigma - 1) \log |\tau| \\
 &\quad - \frac{1}{2}(k + \sigma) \log \left(1 + \frac{\tau^2}{(\sigma + k)^2} \right) - |\tau| \arctan \frac{k + \sigma}{|\tau|}.
 \end{aligned}$$

Now we take $k \sim \tau^2$. Then the second term in the right hand side of (13) equals $2(1 - \sigma) \log |\tau|$ up to the additive constant. The fourth term is bounded. The fifth term equals

$$|\tau| \arctan(|\tau| + o(1)) = \frac{\pi}{2} |\tau| + o(1).$$

So

$$(14) \quad r(k) \leq -\frac{\pi}{2} |\tau| + (1 - \sigma) \log |\tau| + C; k \sim \tau^2.$$

Therefore after substitution $[\tau^2]$ instead of k into (11) we obtain

$$|b(s)| \leq C |\tau|^{2(\sigma-1)} \exp \left(-\frac{\pi}{2} |\tau| + (1 - \sigma) \log |\tau| \right) = C |\tau|^{\sigma-1} \exp \left(-\frac{\pi}{2} |\tau| \right).$$

4. ESTIMATION OF $|\zeta(\sigma + i\tau)|$ FOR LARGE $|\tau|$

Proposition 3. *If operator A satisfies the assumptions of this paper then*

$$|\zeta(\sigma + i\tau)| \leq C |\tau|^{-\sigma-1/2} \exp \left(\frac{\pi}{2} |\tau| \right), \quad C = C(\sigma), |\tau| \geq 1.$$

Proof. We shall use the representation

$$(15) \quad \zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \theta(t) dt = \frac{J(s)}{\Gamma(s)}$$

with

$$(16) \quad \theta(t) = \text{Tr } e^{-tA} = \sum_{j=1}^\infty e^{-t\mu_j}.$$

It is well known (e.g. see [2]) that

$$(17) \quad \theta(t) \sim e^{-at}, \quad t \rightarrow \infty$$

and (Minakshisundaram-Pleijel expansion)

$$(18) \quad \theta(t) \sim \sum_{j=-d}^0 a_j t^{j/n} + \sum_{j=1}^\infty (a_j t^{j/n} + b_j t^{j/n} \log t), \quad t \rightarrow 0.$$

Moreover one can differentiate both expansions (17) and (18). Split the integral from the right hand side of (15):

$$J(s) = J_1(s) + J_2(s) = \int_0^1 t^{s-1} \theta(t) dt + \int_1^\infty t^{s-1} \theta(t) dt.$$

To estimate J_2 we use partial integration

$$\begin{aligned} |J_2(s)| &= \left| \frac{\theta(1)}{s} + \frac{1}{s} \int_1^\infty t^s \theta'(t) dt \right| \leq \frac{1}{|s|} \left[\theta(1) + C \int_1^\infty t^\sigma e^{-at} dt \right] \\ &\leq \frac{C}{|\tau|}; \quad C = C(\sigma). \end{aligned}$$

To estimate J_1 we take $l = 1 + \lceil \sigma \rceil$. Then

$$\theta(t) = \sum_{j=-d}^l (a_j t^{j/n} + b_j t^{j/n} \log t) + r(t)$$

($b_j = 0$ if $j \leq 0$) with

$$|r(t)| \leq C t^{(l+1)/n} |\log t| \quad \text{and} \quad |r'(t)| \leq C t^{l/n} |\log t|; \quad 0 \leq t \leq 1.$$

So

$$\begin{aligned} J_1(s) &= \sum_{j=-d}^l a_j \int_0^1 t^{j/n+s-1} dt + \sum_{j=1}^l b_j \int_0^1 t^{j/n+s-1} \log t dt + \int_0^1 t^{s-1} r(t) dt \\ &= \sum_{j=-d}^l \frac{a_j}{s+j/n} - \sum_{j=1}^l \frac{b_j}{(s+j/n)^2} + \frac{1}{s} r(1) - \frac{1}{s} \int_0^1 t^s r'(t) dt \end{aligned}$$

and

$$|J_1(s)| \leq C/|s| \leq C/|\tau|.$$

Finally

$$(19) \quad |J(s)| \leq \frac{C}{|\tau|}, \quad C = C(\sigma).$$

To estimate $1/\Gamma(s)$ we use the Stirling asymptotics

$$\log \Gamma(s) = (z - 1/2) \log z - z + (1/2) \log(2\pi) + o(1); \quad |z| \rightarrow \infty, |\arg z| < \pi$$

(e.g. see [6]). We can write

$$\begin{aligned} \log \Gamma(\sigma + i\tau) &= [(\sigma - 1/2) + i\tau][(1/2) \log(\sigma^2 + \tau^2) + i \arg(\sigma + i\tau)] \\ &\quad - \sigma - i\tau + (1/2) \log(2\pi) + o(1) \end{aligned}$$

and

$$|\Gamma(\sigma + i\tau)| \sim \sqrt{2\pi} e^{-\sigma} \exp((1/2)(\sigma - 1/2) \log(\sigma^2 + \tau^2) - \tau \arg(\sigma + i\tau)); \quad |\tau| \rightarrow \infty.$$

Note that

$$\arg(\sigma + i\tau) = \frac{\pi}{2} \operatorname{sgn} \tau - \frac{\sigma}{\tau} + o\left(\frac{1}{|\tau|}\right)$$

and

$$\log(\sigma^2 + \tau^2) = 2 \log |\tau| + o(1/|\tau|^2).$$

Therefore

$$(20) \quad |\Gamma(\sigma + i\tau)| \sim \sqrt{2\pi} |\tau|^{\sigma-1/2} e^{-\pi|\tau|/2}; \quad |\tau| \rightarrow \infty.$$

Now the assertion of Proposition 3 follows from (15), (19) and (20).

5. PROOF OF THE THEOREM

Propositions 2 and 3 show us that the function which is integrated in (8) is bounded by $C|\tau|^{-3/2}$. So we can shift the path of integration to the left in the complex plane as far as we want. The poles of the function $b(s)\zeta(s)$ give us terms in asymptotics of $d^*(\omega)$. Denote

$$(21) \quad \begin{aligned} \beta_j &= \text{Res } b(s)|_{s=-j}, \quad j = 0, 1, \dots; \quad \hat{\beta}_0 = \text{Res } sb(s)|_{s=0}; \\ \hat{\beta}_j &= (b(s) - \beta_j/(s+j))|_{s=-j}, \quad j = 1, 2, \dots; \\ \alpha_k &= \text{Res } \zeta(s)|_{s=-k/n}, \quad k = -d, -d+1, \dots; \\ \gamma_j &= (\zeta(s) - \alpha_{nj}/(s+j))|_{s=-j}, \quad j = 1, 2, \dots. \end{aligned}$$

Note that $\alpha_0 = 0$ and if A is a differential operator or it is a power of a differential operator then

$$\alpha_{nj} = 0 \quad \text{and} \quad \gamma_j = \zeta(-j).$$

By Proposition 2

$$\beta_0 = 0 \quad \text{and} \quad \hat{\beta}_0 = 1.$$

The points $s = -k/n$, $k \neq jn$ ($j = 0, 1, \dots$) are simple poles of the function

$$F(s) = e^{s\omega} b(s)\zeta(s)$$

and

$$\text{Res}_{s=-k/n} F(s) = \alpha_k e^{-k\omega/n} b(-k/n).$$

The points $s = jn$ are poles of the second order of $F(s)$. It is easy to calculate residues in these points:

$$\text{Res}_{s=0} F(s) = \zeta'(0) + \omega\zeta(0),$$

$$\text{Res}_{s=-j} F(s) = \alpha_{nj}\beta_j\omega e^{-j\omega} + (\alpha_{nj}\hat{\beta}_j + \gamma_j\beta_j)e^{-j\omega}.$$

Thus we have obtained

$$(22) \quad d^*(\omega) \sim \sum_{k=-d}^{\infty} p_k e^{-k\omega/n} + \sum_{j=0}^{\infty} q_j \omega e^{-j\omega}, \quad \omega \rightarrow \infty$$

with

$$(23) \quad \begin{aligned} p_k &= \alpha_k b(-k/n), \quad k \neq jn \quad (j = 0, 1, \dots); \\ p_0 &= \zeta'(0) \\ p_{jn} &= \alpha_{nj}\hat{\beta}_j + \gamma_j\beta_j, \quad j = 1, 2, \dots; \\ q_0 &= \zeta(0); \quad q_j = \alpha_{nj}\beta_j. \end{aligned}$$

After substituting $e^{-\omega} = \varepsilon$ into (22) one obtains (4). In particular (5) holds.

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