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ON SEMISIMPLE MALCEV ALGEBRAS

ALBERTO ELDUQUE

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ABSTRACT. Let M be a finite dimensional semisimple Malcev algebra over a perfect field of characteristic $\neq 2,3$. Let $N(M)$ be its J-nucleus and $J(M,M,M)$ the subspace spanned by its jacobians. Then it is shown that $M = N(M) \oplus J(M,M,M)$, $N(M)$ is a semisimple Lie algebra and $J(M,M,M)$ is a direct sum of simple non-Lie Malcev algebras.

1. INTRODUCTION

In what follows F will denote always a field of characteristic not two. A Malcev algebra M over F is an anticommutative algebra $(x^2 = 0\ \forall x)$ satisfying the identity

$$(xz)(yt) = ((xy)z)t + ((yz)t)x + ((zt)x)y + ((tx)y)z.$$

Every Lie algebra is a Malcev algebra and many results for Lie algebras have been extended to the Malcev case, see [3], [4], [5], [6], [8], [9], [13], [14], [16], [18] and [19].

If M is a Malcev algebra and $x,y,z \in M$, the element $J(x,y,z) = (xy)z + (yz)x + (zx)y$ is called the jacobian of x,y,z. The subspace spanned by the jacobians is denoted $J(M,M,M)$. Then the subspace $N(M) = \{x \in M: J(x,M,M) = 0\}$ is called the J-nucleus of M.

We shall need the following assertions; their proofs may be found in [16]:

Proposition 1.1. *Let M be a Malcev algebra and $x,y,z,t \in M$. Then:*

(a) $J(x,y,tz) + J(t,y,xz) = J(x,y,z)t + J(t,y,z)x$.

(b) $2tJ(x,y,z) = J(t,x,yz) + J(t,y,zx) + J(t,z,xy)$.

(c) $J(tx,y,z) = tJ(x,y,z) + J(t,y,z)x - 2J(yz,t,x)$.

(d) *If $J(x,y,z) = 0$ then the subalgebra generated by x,y and z is a Lie algebra.*

(e) *$N(M)$ and $J(M,M,M)$ are ideals of M and $N(M)J(M,M,M) = 0$.*

(f) *If $J(x,y,M) = 0$ then $xy \in N(M)$.*

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For any x in the Malcev algebra M, R_x will denote the mapping $y \mapsto yx$. If H is a nilpotent subalgebra and M is finite dimensional, M can be decomposed, as in the Lie case, in primary components. For algebraically closed F, M decomposes as a direct sum of root spaces with respect to the set of endomorphisms $\{R_h : h \in H\}$. If $M_{0H} = H$, H is said to be a Cartan subalgebra. The existence of Cartan subalgebras is guaranteed for solvable Malcev algebras ([5]) and for infinite F ([13]).

Proposition 1.2. *Let M be a finite dimensional Malcev algebra over F, F an algebraically closed field (of characteristic not two). Let H be a nilpotent subalgebra of M and $M = \bigoplus_{\alpha\in\Phi} M_\alpha$ the corresponding decomposition in root spaces. Then:*

(*a*) $M_\alpha M_\beta \subseteq M_{\alpha+\beta}$ *if* $\alpha \neq \beta$.
(*b*) $(M_\alpha)^2 \subseteq M_{2\alpha} + M_{-\alpha}$.
(*c*) $J(M_0, M_0, M_\alpha) = 0$ *if* $\alpha \neq 0$.
(*d*) $J(M_0, M_\alpha, M_\alpha) \subseteq M_{-\alpha}$.
(*e*) $J(M_0, M_\alpha, M_\beta) = 0$ *if* $\alpha \neq \beta$.
(*f*) $J(M_\alpha, M_\beta, M_\gamma) = 0$ *if* $\alpha \neq \beta \neq \gamma \neq \alpha$.
(*g*) $J(M_\alpha, M_\alpha, M_\beta) = 0$ *if* $\beta \neq 0, \pm\alpha$.
(*h*) $J(M_\alpha, M_\alpha, M_{-\alpha}) \subseteq M_\alpha$.
(*i*) $J(M_\alpha, M_\alpha, M_\alpha) \subseteq M_0$.

Proof. (a) to (g) may be found in [2] and [13].

For (h), if $\alpha = 0$ it is clear, if $\alpha \neq 0$ take $h \in H, a_\alpha, b_\alpha \in M_\alpha$ and $a_{-\alpha} \in M_{-\alpha}$. Then, by 1.1 (c) $J(a_\alpha h, b_\alpha, a_{-\alpha}) = a_\alpha J(h, b_\alpha, a_{-a}) + J(a_\alpha, b_\alpha, a_{-\alpha})h - 2J(b_\alpha a_{-\alpha}, a_\alpha, h) = J(a_\alpha, b_\alpha, a_{-\alpha})h$. So $J(\alpha_\alpha, b_\alpha, a_{-\alpha})(R_h - \alpha(h)1)^n = J(a_\alpha(R_h - \alpha(h)1)^n, b_\alpha, a_{-\alpha}) = 0$ for large enough n.

For (i), first notice that $J(M_\alpha, M_\alpha, M_\alpha) \subseteq (M_\alpha)^2 M_\alpha \subseteq M_{3\alpha} + M_0$, so if the characteristic is 3 we are done. In other case take $a_\alpha, b_\alpha, c_\alpha \in M_\alpha$ and $h \in H$. By 1.1 (c) $J(a_\alpha h, b_\alpha, c_\alpha) = a_\alpha J(h, b_\alpha, c_\alpha) + J(a_\alpha, b_\alpha, c_\alpha)h - 2J(b_\alpha c_\alpha, a_\alpha, h) = a_\alpha J(h, b_\alpha, c_\alpha) + J(a_\alpha, b_\alpha, c_\alpha)h \in J(a_\alpha, b_\alpha, c_\alpha)h + M_0$. Hence, for large enough n, $J(a_\alpha, b_\alpha, c_\alpha)(R_h - \alpha(h)1)^n \in M_0$, so $J(a_\alpha, b_\alpha, c_\alpha) \in M_\alpha + M_0$. Then $J(M_\alpha, M_\alpha, M_\alpha) \subseteq (M_{3\alpha} + M_0) \cap (M_\alpha + M_0) = M_0$.

Every simple Malcev algebra is either a Lie algebra or a seven dimensional algebra over its centroid, obtained as the set of elements of trace 0 of a Cayley-Dickson algebra under commutation (see [13]). In any finite dimensional Malcev algebra M, the largest solvable ideal will be denoted by $R(M)$. If $R(M) = 0$, M is said to be semisimple. In the characteristic 0 case any semisimple Malcev algebra is a direct sum of simple algebras ([13, Theorem 8]). Of course, this is not valid for prime characteristic since it does not work even for Lie algebras. If the characteristic is 3, Filippov proved in [11] that any semisimple Malcev algebra is a Lie algebra.

The aim of this paper is to show that, at least, the 'non-Lie part' of a semisimple Malcev algebra over a perfect field is a direct sum of simple non-Lie algebras, and that M decomposes as a 'Lie part' and a 'non-Lie part'. More precisely the following result will be proved:

'*Any finite dimensional semisimple Malcev algebra over a perfect field decomposes as*

$$M = N(M) \oplus J(M, M, M);$$

$N(M)$ is a semisimple Lie algebra and $J(M, M, M)$ is a direct sum of simple non-Lie Malcev algebras'.

This will be done in Section 4. In Sections 2 and 3 we shall study Malcev algebras with a trivial J-nucleus and obtain some results needed for the Theorem above.

Our results will cover some of the results in [7], where Malcev algebras of prime characteristic have been studied following the ideas related to restricted Lie algebras. We will see in the paper that the restrictions these ideas impose on the Malcev algebras are not necessary.

All the algebras considered from now on will be supposed to be finite dimensional.

2. Malcev algebras with trivial J-nucleus

In this section, F will denote an algebraically closed field of characteristic not two. The only, up to isomorphism, simple non-Lie Malcev algebra over F will be denoted by C_0.

Let M be a finite dimensional Malcev algebra over F with $N(M) = 0$ (so actually the characteristic of F is not 3 [10]). Let H be a Cartan subalgebra of M and $M = \bigoplus_{\alpha\in\Phi} M_\alpha$ be the corresponding root space decomposition. Then:

Proposition 2.1.

(*a*) *If* $\alpha \in \Phi$ *then* $-\alpha \in \Phi$.

(*b*) $(M_\alpha)^2 \subseteq M_{-\alpha}$ *for all* $\alpha \in \Phi$.

(*c*) $M_\alpha M_\beta = 0$ *if* $\beta \neq 0, \pm\alpha$.

(*d*) H *is abelian and for all* x, y *in* H, R_x *and* R_y *commute.*

(*e*) *Each* $\alpha \in \Phi$ *is a linear function on* H.

Proof. For (a) let us suppose that $M_\alpha \neq 0$ but $M_{-\alpha} = 0$. Then by 1.2 (c), (d) and (e) we have $J(H, M_\alpha, M) = 0$, so by 1.1 f) $M_\alpha = HM_\alpha \subseteq N(M) = 0$, a contradiction.

Assertions (b) and (c) may be found in [10, proof of Theorem 3.5].

For (d) notice that $J(H, H, M_\alpha) = 0$ and $J(H, H, H)M_\alpha = 0$, because of 1.1 (b). Hence $J(H, H, H)$ is an ideal of M which annihilates $\bigoplus_{\alpha\neq0} M_\alpha$. But H is nilpotent so, if $J(H, H, H) \neq 0$, there is an element $0 \neq h \in J(H, H, H)$ such that $hH = 0$. Then $hM = 0$ and $h \in N(M) = 0$, a contradiction. We therefore have $J(H, H, M) = 0$, thus $H^2 = 0$ and $[R_x, R_y] = 0$ for all x, y in H (this is also proved in [10], but the proof above is easier).

Assertion (e) follows easily.

For $\alpha \in \Phi - 0$ let us consider the subspace $S^\alpha = M_\alpha M_{-\alpha} \oplus M_\alpha \oplus M_{-\alpha}$; the Proposition above shows that S^α is an ideal of M and that $S^\alpha S^\beta = 0$ if $\beta \neq 0, \pm\alpha$.

Let us take a system of representatives Φ^+ of the sets $\{\alpha, -\alpha\}$ with $0 \neq \alpha \in \Phi$ and form the subalgebra $\sum_{\alpha \in \Phi^+} S^\alpha$.

Proposition 2.2. *The sum* $\sum_{\alpha\in\Phi^+} S^\alpha$ *is direct.*

Proof. Let $\alpha, \beta_1, \dots, \beta_r \in \Phi^+$ and $x \in S^\alpha \cap (S^{\beta_1} + \cdots + S^{\beta_r})$. Then $x \in H$ so $J(x, M_\mu, M_\nu) = 0$ if either $\mu = 0$, $\nu = 0$ or $\mu \neq \nu$. Now, if $\mu \neq 0$, $J(x, M_\mu, M_\mu) \subseteq M_{-\mu} \cap S^\alpha \cap (S^{\beta_1} + \cdots + S^{\beta_r}) \subseteq M_{-\mu} \cap H = 0$. Hence $J(x, M, M) = 0$ and $x = 0$.

Now, if we take a complementary subspace H' in H to $\oplus_{\alpha\in\Phi^+} M_\alpha M_{-\alpha}$ we have the decomposition

$$M = H' \oplus \left(\bigoplus_{\alpha\in\Phi^+} S^\alpha \right).$$

Let us study the ideals S^α.

Proposition 2.3. *If* $\alpha \in \Phi^+$, *then* $\alpha(M_\alpha M_{-\alpha}) \neq 0$, *the solvable radical* $R(S^\alpha)$ *is the unique maximal ideal of* S^α, $S^\alpha / R(S^\alpha)$ *is, therefore, simple, and* $R(S^\alpha)$ *is nilpotent.*

Proof. Let us suppose that $\alpha(M_\alpha M_{-\alpha}) = 0$. If $x_\alpha \in M_\alpha$, $u \in H$ and $h \in H$ with $\alpha(h) \neq 0$, then $J(h, u, x_\alpha) = 0$ so, by 1.1 d), the subalgebra T generated by $\{h, u, x_\alpha\}$ is a Lie algebra and its corresponding root space decomposition with respect to R_h is $T = T_0 \oplus T_\alpha \oplus T_{-\alpha}$. Then $(ux_\alpha)x_\alpha \in T_{2\alpha} = 0$, so $H(R_{x_\alpha})^2 = 0$. Hence $M_{-\alpha}(R_{x_\alpha})^3 = 0$ and $M_\alpha(R_{x_\alpha})^4 = 0$.

We therefore have that for all $x \in M_\alpha M_{-\alpha} \cup M_\alpha \cup M_{-\alpha}$, R_{x_α} is nilpotent.

Because of [18, Corollary 1], S^α is nilpotent.

Let $x = x_0 + x_\alpha + x_{-\alpha} \in S^\alpha$ be a nonzero element with $xS^\alpha = 0$ where $x_0 \in M_\alpha M_{-\alpha}$, $x_{\pm\alpha} \in M_{\pm\alpha}$. Then $x_0 S^\alpha = x_\alpha S^\alpha = x_{-\alpha} S^\alpha = 0$, so $x_0 M = 0$ and $x_0 \in N(M) = 0$. Without loss of generality we may suppose that $x_\alpha \neq 0$. Now $J(x_\alpha, M, M) = J(x_\alpha, H, M_\alpha)$ because of 1.2. But if $h \in H$ and $y_\alpha \in M_\alpha$, then $(h(x_\alpha + y_\alpha))(x_\alpha + y_\alpha) = 0$ so $(hx_\alpha)y_\alpha = -(hy_\alpha)x_\alpha = 0$ and $J(x_\alpha, h, y_\alpha) = (x_\alpha h)y_\alpha + (hy_\alpha)x_\alpha + (y_\alpha x_\alpha)h = 0$. Hence $x_\alpha \in N(M) = 0$, a contradiction which proves that $\alpha(M_\alpha M_{-\alpha}) \neq 0$.

This implies that $(S^\alpha)^2 = S^\alpha$ so S^α is not solvable. If I is an ideal of S^α then I is $(M_\alpha M_{-\alpha})$-invariant, so $I = I \cap (M_\alpha M_{-\alpha}) \oplus I \cap M_\alpha \oplus I \cap M_{-\alpha}$. If $\alpha(I \cap (M_\alpha M_{-\alpha})) \neq 0$, then $M_\alpha + M_{-\alpha} \subseteq I$ and $I = S^\alpha$. In other case I is shown to be nilpotent as above and the remaining assertions of the Proposition follow.

Corollary 2.4 ([10]). *If* M *is solvable then* $N(M) \neq 0$.

The next Lemma is known, it follows from some deep results of Lie theory (see [21]). We include a proof for completeness:

Lemma 2.5. *Let L be a simple Lie algebra with an abelian Cartan subalgebra H such that the root space decomposition is $L = H \oplus L_\alpha \oplus L_{-\alpha}$ and α is linear on H. Then L is isomorphic to $sl(2,F)$ (the Lie algebra of traceless 2×2 matrices under commutation).*

Proof. Let $T = \operatorname{Ker}\alpha \oplus (\operatorname{Ker}\alpha)L_\alpha \oplus (\operatorname{Ker}\alpha)L_{-\alpha}$. T is obviously H-invariant. Now, let $0 \neq h \in \operatorname{Ker}\alpha, 0 \neq x \in L_\alpha$ and $0 \neq y \in L_{-\alpha}$ and take m such that $y_1 = y(R_h)^{(m-1)} \neq 0$, $y_1 h = 0$. Then $((hx)y)y_1 = ((hx)y_1)y = ((hy_1)x)y = 0$ because $H^2 = (L_{\pm\alpha})^2 = 0$, so $(hx)y \in \operatorname{Ker}\alpha$. We therefore have $((\operatorname{Ker}\alpha)L_\alpha)L_{-\alpha} \subseteq \operatorname{Ker}\alpha$. Hence T is an ideal of L so $T = 0$ and H has dimension 1.

Let us take now $x \in L_\alpha$, $y \in L_{-\alpha}$ with $xy \neq 0$. For all $z \neq 0$ in L_α, $(zy)x = z(yx) \neq 0$, so the mapping $R_y : L_\alpha \to H$ is injective. Hence L_α, and $L_{-\alpha}$ too, has dimension 1, and the Lemma follows.

Corollary 2.6. *Let M be a Malcev algebra over F with $N(M) = 0$. Then $R(M^2)$ is nilpotent and $M^2/R(M^2)$ is a direct sum of simple algebras isomorphic either to $sl(2,F)$ or to C_0.*

Proposition 2.7. *For any α in Φ^+, $N(S^\alpha) = 0$.*

Proof. We can pick up an element $h \in M_\alpha M_{-\alpha}$ with $\alpha(h) = 1$. If $0 \neq x_0 + x_\alpha + x_{-\alpha} \in N(S^\alpha)$, where $x_0 \in M_\alpha M_{-\alpha}$, $x_{\pm\alpha} \in M_{\pm\alpha}$, then x_0, x_α and $x_{-\alpha}$ are in $N(S^\alpha)$. Then $x_0 \in N(M) = 0$. So we may suppose that $0 \neq x_\alpha \in N(S^\alpha)$, $x_\alpha h = x_\alpha$ and $x_\alpha(\operatorname{Ker}\alpha \cap M_\alpha M_{-\alpha}) = 0$. Then $x_\alpha M_{-\alpha} \subseteq N(S^\alpha) \cap H \subseteq N(M) = 0$, and if $y_\alpha \in M_\alpha$, the subalgebra T generated by $\{h, x_\alpha, y_\alpha\}$ is a Lie algebra so $x_\alpha y_\alpha \in T_{2\alpha} = 0$. Hence $x_\alpha(M_\alpha + M_{-\alpha}) = 0$ and $x_\alpha S^\alpha = 0$, a contradiction with $x_\alpha h = x_\alpha$.

Now, if $\alpha \in \Phi^+$ and $S^\alpha/R(S^\alpha) \cong C_0$ then by [4] there is a subalgebra $A^\alpha \cong C_0$ such that $S^\alpha = R(S^\alpha) \oplus A^\alpha$. The same happens if $S^\alpha/R(S^\alpha) \cong sl(2,F)$:

Proposition 2.8. *Let M be a Malcev algebra with $N(M) = 0$ and $M/R(M) \cong sl(2,F)$. Then there is a subalgebra S such that $M = R(M) \oplus S$.*

Proof. Let H be any Cartan subalgebra of M. Then $M = H \oplus M_\alpha \oplus M_{-\alpha}$. Let T be a minimal subalgebra with the condition $M = R(M) + T$. Then $R(M) \cap T$ is contained in the Frattini ideal $\varphi(T)$ of T (see [20, Lemma 7.1]). But $(T/(R(M)\cap T)) \cong sl(2,F)$, so $R(M)\cap T = \varphi(T) = R(T)$. If $R(M)\cap T = 0$ we are done. In the other case, let $h \in T - \varphi(T)$ with non-nilpotent R_h. The Fitting null component with respect to R_h, $M_0(h) = Fh + (M_0(h) \cap R(M))$, is a nilpotent subalgebra of M, since $R(M)$ is nilpotent, so it is a Cartan subalgebra, $M = M_0(h) \oplus M_\alpha(h) \oplus M_{-\alpha}(h)$ and $T = T_0(h) \oplus T_\alpha(h) \oplus T_{-\alpha}(h)$.

Let B be an ideal of T such that $\varphi(T)/B$ is a composition factor of T. Then $\varphi(T)/B$ is an irreducible module for $T/\varphi(T) \cong sl(2,F)$, and since the only possible weight spaces that appear are $0, \alpha, -\alpha$, and the characteristic is $\neq 2, 3$, there are only three possibilities (see [2] and [15]):

—A trivial module of dimension 1.

—A non-Lie Malcev module of dimension 2.

—A regular module of dimension 3.

In these three cases, the reasoning in [4, p. 182] shows that there is a subalgebra S of T such that $T/B = (\varphi(T)/B) \oplus (S/B) = \varphi(T/B) \oplus (S/B)$. This is a contradiction with [20, Lemma 2.1]. Hence $M = R(M) \oplus T$.

The result above was proved in [7] with more stringent conditions.

Gathering together the results in this section we get:

Theorem 2.9. *Let* M *be a Malcev algebra over* F *with* $N(M) = 0$ *and let* H *be any Cartan subalgebra of* M. *If* $M = \oplus_{\alpha\in\Phi} M_\alpha$ *is the corresponding root space decomposition then the* α*'s are linear;* $H^2 = 0$*; for any* $\alpha \neq 0$, $S^\alpha = M_\alpha M_{-\alpha} \oplus M_\alpha \oplus M_{-\alpha}$ *is an ideal and*

$$M = H' \oplus \left(\bigoplus_{\alpha\in\Phi^+} S^\alpha \right)$$

where $H' \subseteq \{x \in H : \alpha(x) = 0\ \forall\alpha \in \Phi^+\}$, Φ^+ *is a system of representatives of the sets* $\{\alpha, -\alpha\}$ *where* $\alpha \in \Phi - 0$ *and each* S^α *decomposes as* $S^\alpha = R(S^\alpha) \oplus A^\alpha$, *with* $R(S^\alpha)$ *nilpotent and* A^α *isomorphic either to* $sl(2, F)$ *or to* C_0.

3. Semisimple Malcev algebras with trivial J-nucleus

In this section F will denote an algebraically closed field of prime characteristic $\neq 2, 3$.

Let us recall the definition of 'quasiderivation' given by R. Block, restricted to Malcev algebras. Given a Malcev algebra M and a linear mapping $d: M \to M$, d is called a *quasiderivation* if $[d, R_x] \subseteq M^+$ (M^+ is the multiplication algebra of M; that is, the associative subalgebra of $\mathrm{End}_F(M)$ generated by $\{R_x : x \in M\}$). The algebra M is said to be *quasidifferentiably simple* if it does not contain any proper ideal invariant under quasiderivations.

In the remainder of the section M will denote a semisimple Malcev algebra over F with $N(M) = 0$ and we shall use the same notation as in the last section with $H' \subseteq \{x \in H : \alpha(x) = 0\ \forall\alpha \in \Phi\}$.

Lemma 3.1. *For any* x *in* H' *and* α *in* Φ^+, *the restriction of* R_x *to* S^α *is a quasiderivation of* S^α.

Proof. Take $h \in M_\alpha M_{-\alpha}$ with $\alpha(h) = 1$, $z \in S^\alpha$ and $y_\alpha \in M_\alpha$. There is an $u_\alpha \in M_\alpha$ with $y_\alpha = u_\alpha h$.

Then $J(z, y_\alpha, x) = J(z, u_\alpha h, x) = 2zJ(u_\alpha, h, x) - J(z, hx, u_\alpha) - J(z, xu_\alpha, h) = -J(z, xu_\alpha, h)$, so $[R_{y_\alpha}, R_x] = R_{(y_\alpha x)} - [R_{(xu_\alpha)}, R_h] + R_{(xu_\alpha)h}$.

But $y_\alpha x, xu_\alpha$ and $(xu_\alpha)h$ are in S^α, so the restriction of $[R_{y_\alpha}, R_x]$ to S^α belongs to $(S^\alpha)^+$.

The same for $y_{-\alpha} \in M_{-\alpha}$ and, finally, if $y_0 \in M_\alpha M_{-\alpha}$, $[R_{y_0}, R_x] = 0$ by 2.1 (d).

Lemma 3.2. *For any* $\alpha \in \Phi^+$, S^α *is quasidifferentiably simple.*

Proof. Let B be an ideal of S^α, $B \neq S^\alpha$, invariant under R_x, for all $x \in H'$. By the last section B is nilpotent and B is an ideal of the whole M, but M is semisimple, so $B = 0$.

The next proposition shows that the case $A^\alpha \cong sl(2, F)$ is not possible:

Proposition 3.3. *For any* $\alpha \in \Phi^+$, A^α *is isomorphic to* C_0.

Proof. Let us suppose that there is an $\alpha \in \Phi^+$ with $A^\alpha \cong sl(2, F)$. Then $R(S^\alpha) \neq 0$ since $N(S^\alpha) = 0$. By [1, Lemma 2.2] and the last lemma, any composition factor of S^α is isomorphic, as a $(S^\alpha)^+$-module, to any minimal ideal of S^α. Now, if B is a minimal ideal of S^α, then $B \subseteq R(S^\alpha)$ and the nilpotency of $R(S^\alpha)$ shows that $BR(S^\alpha) = 0$. Hence B is an irreducible non-Lie $(N(S^\alpha) = 0)$ Malcev module for $A^\alpha \cong sl(2, F)$. By [2] the dimension of B is 2. But the dimension of the composition factor $S^\alpha / R(S^\alpha)$ is 3, a contradiction.

If $\alpha \in \Phi^+$, by the result above $S^\alpha / R(S^\alpha) \cong C_0$. By [4], we have that the exact sequence of algebras and morphisms of algebras

$$0 \to R(S^\alpha) \to S^\alpha \to S^\alpha / R(S^\alpha) \to 0$$

splits.

Now, [1, Lemmas 3.3 and 4.1] imply that $S^\alpha \cong A^\alpha \otimes_F B_{n_\alpha}(F)$, for some natural number n_α, where $B_{n_\alpha}(F)$ is the p-truncated polynomial algebra $F[X_1, \dots, X_n]/(X_1^p, \dots, X_n^p)$, p being the characteristic of F, and $A^\alpha \cong C_0$.

We shall denote by x_i the class of X_i modulo the ideal $(X_1^p, \dots, X_n^p)$. For each $\alpha \in \Phi^+$ we shall identify S^α with $A^\alpha \otimes_F B_{n_\alpha}(F)$. Let $\{h^\alpha, e_1^\alpha, e_2^\alpha, e_3^\alpha, f_1^\alpha, f_2^\alpha, f_3^\alpha\}$ be a Sagle basis of A^α as in [8, p. 220].

Then $H^\alpha = Fh^\alpha \otimes_F B_{n_\alpha}(F)$ is a Cartan subalgebra of S^α and $G = \bigoplus_{\alpha \in \Phi^+} H^\alpha$ is an abelian subalgebra of M^2. If $M = M_{0G} \oplus M_{1G}$ is the Fitting decomposition of M with respect to G, then $J(M_{0G}, M_{0G}, M_{0G})$ turns out to be an ideal of M (same reasoning as in 2.1) contained in $M_{0G} \cap M^2 = G$. By semisimplicity $J(M_{0G}, M_{0G}, M) = J(M_{0G}, M_{0G}, M_{0G}) = 0$ and $(M_{0G})^2 = 0$. Thus M_{0G} is a Cartan subalgebra of M.

The conclusion of this is that we may take the Cartan subalgebra H at the begininning with $H \cap S^\alpha = H^\alpha$ for all $\alpha \in \Phi^+$. We do so.

Let us pick elements $z \in H'$ and $v \in Fe_1^\alpha + Fe_2^\alpha + Fe_3^\alpha$. The subalgebra T generated by $v \otimes 1$, $h^\alpha \otimes 1$ and z is a Lie algebra and, as in 2.3, we get

$(v \otimes 1)((v \otimes 1)z) = 0$. Hence $(v \otimes 1)z \in M_\alpha \cap \{b \in S^\alpha : (v \otimes 1)b = 0\} = Fv \otimes_F B_{n_\alpha}(F)$. Then there exists a $p_v(x) \in B_{n_\alpha}(F)$ such that $(v \otimes 1)z = v \otimes p_v(x)$ $(x = (x_1, \dots, x_n))$. From $(v \otimes 1)z = v \otimes p_v(x)$, $(w \otimes 1)z = w \otimes p_w(x)$ and $((v + w) \otimes 1)z = (v + w) \otimes p_{v+w}(x)$, we get that $p_v(x) = p_w(x)$ for all $v, w \in Fe_1^\alpha + Fe_2^\alpha + Fe_3^\alpha$ and this common element in $B_{n_\alpha}(F)$ will be denoted by $p(x)$.

In the same way there is a $q(x) \in B_{n_\alpha}(F)$ such that for all u in $Ff_1^\alpha + Ff_2^\alpha + Ff_3^\alpha$, we have $(u \otimes 1)z = u \otimes q(x)$.

Since $e_1^\alpha f_1^\alpha = (1/2)h^\alpha$, $J(e_1^\alpha \otimes 1, f_1^\alpha \otimes 1, z) = 0$ and $(h^\alpha \otimes 1)z = 0$ we get that $h^\alpha \otimes (p(x) + q(x)) = 0$, so $p(x) = -q(x)$.

We therefore have that $z - (h^\alpha \otimes p(x))$ annihilates S^α, so that we can take H' such that $H'S^\alpha = 0$. Proceeding in the same way with all $\beta \in \Phi^+$ we conclude that H' may be chosen with $H'M^2 = 0$ and, since M is semisimple, this implies that $H' = 0$ and each $R(S^\alpha)$ equals 0. Hence:

Theorem 3.4. *Let M be a semisimple Malcev algebra over an algebraically closed field F with $N(M) = 0$. Then M is a direct sum of ideals which are isomorphic to C_0.*

4. Semisimple Malcev algebras

Let us define $N_1(M) = N(M)$ and $N_{i+1}(M)$ by means of

$$N_{i+1}(M)/N_i(M) = N(M/N_i(M)).$$

It is known ([10]) that if M is solvable then there is an $r \in \mathbf{N}$ such that $M = N_r(M)$. The situation for semisimple Malcev algebras is quite different:

Proposition 4.1. *If M is a semisimple Malcev algebra, then $N_2(M) = N(M)$ (so for all $r, N_r(M) = N(M)$).*

Proof. $N(M)J(M, M, M) = 0$ (1.1e)) so $N(M) \cap J(M, M, M) = 0$, since M is semisimple. Now, $J(N_2(M), M, M) \subseteq J(M, M, M) \cap N(M) = 0$, so $N_2(M) \subseteq N(M)$.

Proposition 4.2. *If M is a semisimple Malcev algebra, then $M/N(M)$ is also semisimple.*

Proof. Let B be an ideal of M such that $B/N(M)$ is a minimal abelian ideal of $M/N(M)$. Then $J(B, M, M)$ is not contained in $N(M)$ because of 4.1, so $N(M) \subset J(B, M, M) + N(M) \subseteq B$. Hence $B = J(B, M, M) \oplus N(M)$. But, in this case, $J(B, M, M)$ would be an abelian ideal of M, a contradiction.

Theorem 4.3. *Let M be a semisimple Malcev algebra over an algebraically closed field F. Then $M = N(M) \oplus J(M, M, M)$, where the J-nucleus $N(M)$ is a semisimple Lie algebra and the ideal $J(M, M, M)$ is a direct sum of copies of C_0.*

Proof. By 4.1, 4.2 and 3.4 we have that $J(M/N(M), M/N(M), M/N(M)) = M/N(M)$, so $M = N(M) + J(M, M, M)$ and, as shown above, $N(M) \cap J(M, M, M) = 0$.

Notice that in case $\operatorname{char} F = 3$, as shown by Filippov [11], any semisimple Malcev algebra is a Lie algebra, so 4.3 is trivial in this case.

Now, reasoning as in [17, Chapter V, §6], one gets that if F is a perfect field, Ω an algebraic closure of F and M a semisimple Malcev algebra over F, then $M_\Omega = M \otimes_F \Omega$ is semisimple. For this, the only thing needed is the following lemma:

Lemma 4.4. *Let B be an ideal of a Malcev algebra M with $B^2 = B$. Then $C_M(B) = \{x \in M : xB = 0\}$ is an ideal of M.*

Proof. Let $x, z \in B$, $y \in C_M(B)$ and $t \in M$; then $(xz)(yt) = ((xy)z)t + ((yz)t)x + ((zt)x)y + ((tx)y)z = 0$, so $(yt)B^2 = (yt)B = 0$.

Hence, if M is a semisimple Malcev algebra over a perfect field F, then $M_\Omega = N(M_\Omega) \oplus J(M_\Omega, M_\Omega, M_\Omega)$. But $N(M_\Omega) = N(M) \otimes_F \Omega$ ([10, Proposition 3.4]), and obviously $J(M_\Omega, M_\Omega, M_\Omega) = J(M, M, M) \otimes_F \Omega$, so that $M = N(M) \oplus J(M, M, M)$. Moreover, $J(M, M, M) \otimes_F \Omega$ is completely reducible as a module for $(M_\Omega)^+ = (M^+)_\Omega$. Then the same happens for $J(M, M, M)$ so this is a direct sum of simple ideals. In consequence we get:

Theorem 4.5. *Let M be a semisimple Malcev algebra over a perfect field F. Then $M = N(M) \oplus J(M, M, M)$, where the J-nucleus $N(M)$ is a semisimple Lie algebra and the ideal $J(M, M, M)$ is a direct sum of simple non-Lie Malcev algebras* (*seven dimensional over their centroids*).

In [7, Theorem 3.4] it is proved that if M is a weakly restricted semisimple Malcev algebra of toral rank one, with a maximal subalgebra which is solvable, then M is a Lie algebra. These hypotheses may be weakened:

Corollary 4.6. *Let M be a semisimple Malcev algebra over a perfect field F and let S be a maximal subalgebra of M. If S is solvable, then M is a Lie algebra.*

Proof. If $J(M, M, M) \neq 0$ then $M = J(M, M, M) + S$ so $N(M) \cong M/J(M, M, M)$ would be solvable, hence equal to 0. Now if B is a proper ideal of $J(M, M, M)$, $M = B + S$ and M/B would be solvable. Thus M would be simple and the assertion follows from [8, Theorem 4.1].

References

1. R. E. Block, *Determination of the differentiably simple rings with a minimal ideal*, Ann. Math. **90** (1969), no. 3, 433–459.
2. R. Carlsson, *Malcev-Moduln*, J. Reine Angew. Math. **281** (1976), 199–210.
3. ——, *The first Whitehead lemma for Malcev algebras*, Proc. Amer. Math. Soc. **58** (1976), 79–84.
4. ——, *On the exceptional central simple non-Lie Malcev algebras*, Trans. Amer. Math. Soc. **244** (1978), 173–184.
5. A. A. El Malek, *On the Frattini subalgebra of a Malcev algebra*, Arch. Math. **37** (1981), 306–315.
6. ——, *The conjugacy of Cartan subalgebras of a solvable Malcev algebra*, J. Algebra **85** (1983), 323–332.

7. A. A. El Malek and A. Micali, *Algèbres de Malcev faiblement restreintes*, Algebras, Groups and Geometries **3** (1986), 414–430.
8. A. Elduque, *On maximal subalgebras of central simple Malcev algebras*, J. Algebra **103** (1986), 216–227.
9. ——, *On supersolvable Malcev algebras*, Commun. Algebra **14** (1986), 311–321.
10. A. Elduque and A. A. El Malek, *On the J-nucleus of a Malcev algebra*, Algebras, Groups and Geometries **3** (1986), 493–503.
11. V. T. Filippov, *Semiprimary Malcev algebras of characteristic* 3, Algebra and Logic **14** (1975), 64–71.
12. N. Jacobson, *Lie algebras*, Interscience, New York, 1962.
13. E. N. Kuzmin, *Malcev algebras and their representations*, Algebra and Logic **7** (1968), 233–244.
14. ——, *Levi's theorem for Malcev algebras*, Algebra and Logic **16** (1977), 286–291.
15. A. N. Rudakov and I. R. Safarevic, *Irreducible representations of a simple three-dimensional Lie algebra over a field of finite characteristic*, Math. Notes Acad. Sci. USSR **2** (1967), 760–767.
16. A. A. Sagle, *Malcev algebras*, Trans. Amer. Math. Soc. **101** (1961), 426–458.
17. G. B. Seligman, *Modular Lie algebras*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 40, Springer-Verlag, New York, 1967.
18. E. L Stitzinger, *On nilpotent and solvable Malcev algebras*, Proc. Amer. Math. Soc. **92** (1984), 157–163.
19. ——, *Supersolvable Malcev algebras*, J. Algebra **103** (1986), 69–79.
20. D. A. Towers, *A Frattini theory for algebras*, Proc. London Math. Soc. (3) **27** (1973), 440–462.
21. R. L. Wilson, *Simple Lie algebras of toral rank one*, Trans. Amer. Soc. **236** (1978), 287–295.

Departamento de Matemática Aplicada, E.T.S.I.I. Universidad de Zaragoza, 50015 Zaragoza, Spain