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# COMPLEMENTED COPIES OF $c_0$ IN VECTOR-VALUED HARDY SPACES

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ABSTRACT. Let $X$ be a complex Banach space containing a copy of $c_0$, let $\mathbb{T}$ be the unit circle and let $D$ be the open unit disk in the complex plane. Then $H^p(\mathbb{T}, X)$ contains a complemented copy of $c_0$ for $1 \leq p < \infty$. The corresponding result for $H^p(D, X)$ fails for $1 \leq p \leq \infty$.

## 1. INTRODUCTION

If $X$ is a Banach space which contains a copy of $c_0$ then $L^p([0,1], X)$ contains a complemented copy of $c_0$ for $1 \leq p < \infty$ [5]. In this note we consider the corresponding problem for vector-valued Hardy spaces. However, there are two natural Hardy spaces to consider, $H^p(\mathbb{T}, X)$ and $H^p(D, X)$. We will show that $H^p(\mathbb{T}, X)$ contains a complemented copy of $c_0$ whenever $1 \leq p < \infty$ and $X$ is a complex Banach space containing a copy of $c_0$. The proof will allow us to extend the result to a slightly larger class of spaces. We will also show that the spaces $H^p(D, \ell_\infty)$ do not contain complemented copies of $c_0$ for $1 \leq p \leq \infty$.

## 2. PRELIMINARIES AND RESULTS

Throughout this note $\mathbb{T}$ will denote the unit circle, $\frac{d\theta}{2\pi}$ will denote normalized Lebesgue measure on $\mathbb{T}$, and $D$ will be the open unit disk in the complex plane.

Let $X$ be a complex Banach space and let $1 \leq p \leq \infty$. The space $H^p(D, X)$ is the collection of all $X$-valued analytic functions on $D$ with $\|f\|_p < \infty$ where

$$\|f\|_p = \sup_{0 \leq r < 1} \left\{ \int_0^{2\pi} \|f(re^{i\theta})\|^p \frac{d\theta}{2\pi} \right\}^{1/p}$$

for $1 \leq p < \infty$, and

$$\|f\|_\infty = \sup_{z \in D} \|f(z)\|.$$

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If $f\colon \mathbb{T} \to X$ is a vector-valued function then its Fourier coefficients are

$$\hat{f}(n) = \int_0^{2\pi} f(e^{i\theta})e^{-in\theta}\frac{d\theta}{2\pi}, \quad \text{for each } n \in \mathbb{Z}.$$

For $1 \le p \le \infty$, we define

$$H^p(\mathbb{T}, X) = \{f \in L^p(\mathbb{T}, X)\colon \hat{f}(n) = 0 \quad \text{for all } n < 0\}.$$

Before we get to the main result we need a lemma which appears implicitly in [2] and [7].

**Lemma.** *If a Banach space $X$ contains a sequence $(x_n)_{n=1}^\infty$ which is equivalent to the unit vector basis of $c_0$ and if $(x_n^*)_{n=1}^\infty$ is a weak* null sequence in $X^*$ such that $\inf_n |x_n^*(x_n)| > 0$, then $X$ contains a complemented copy of $c_0$.*

*Proof.* Define an operator $S\colon X \to c_0$ by $S(x) = (x_n^*(x))_{n=1}^\infty$. Clearly, $S$ is well defined, since $(x_n^*)_{n=1}^\infty$ is weak* null, and also bounded and linear. The series $\sum_{n=1}^\infty x_n$ is weakly unconditionally Cauchy but $\sum_{n=1}^\infty S(x_n)$ is not unconditionally convergent in $c_0$ because $\inf_n |x_n^*(x_n)| > 0$. By [1] there is a subsequence $(y_n)_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that $(y_n)_{n=1}^\infty$ is equivalent to the unit vector basis of $c_0$ and $S|_{[y_n]_{n=1}^\infty}$ is an isomorphism of $[y_n]_{n=1}^\infty$ onto $Y = [S(y_n)]_{n=1}^\infty$. $Y$ is a subspace of $c_0$ which is isomorphic to $c_0$ and so is complemented in $c_0$ by a bounded linear projection $Q$ (see [8]). Consider the operator $P\colon X \to X$ defined by $P(x) = (S|_{[y_n]_{n=1}^\infty})^{-1}QS(x)$ for $x \in X$. $P$ is a bounded linear projection of $X$ onto $[y_n]_{n=1}^\infty$, and since $[y_n]_{n=1}^\infty$ is isomorphic to $c_0$, the proof is complete.

**Theorem.** *Let $X$ be a complex Banach space and $1 \le p < \infty$. If $X$ contains a copy of $c_0$, then $H^p(\mathbb{T}, X)$ contains a complemented copy of $c_0$.*

*Proof.* Let $(x_n)_{n=1}^\infty$ be a sequence in $X$ equivalent to the unit vector basis of $c_0$. Then there are constants $K_1$, $K_2 > 0$ so that for any choice of scalars $a_1, a_2, \dots, a_n$,

$$K_1 \max_{1\le j\le n} |a_j| \le \|\sum_{j=1}^n a_j x_j\| \le K_2 \max_{1\le j\le n} |a_j|.$$

For each $n \in \mathbb{N}$ define $y_n \in H^p(\mathbb{T})$ by $y_n(e^{i\theta}) = e^{in\theta}$. Then $x_n \otimes y_n \in H^p(\mathbb{T}, X)$, where $(x_n \otimes y_n)(e^{i\theta}) = x_n e^{in\theta}$ and

$$K_1 \max_{1\le j\le n} |a_j| \le \|\sum_{j=1}^n a_j (x_j \otimes y_j)(e^{i\theta})\| \le K_2 \max_{1\le j\le n} |a_j|.$$

Therefore

$$K_1 \max_{1\le j\le n} |a_j| \le \|\sum_{j=1}^n a_j (x_j \otimes y_j)\|_p \le K_2 \max_{1\le j\le n} |a_j|.$$

That is, $(x_n \otimes y_n)_{n=1}^\infty$ is equivalent to the unit vector basis of $c_0$ in $H^p(\mathbb{T}, X)$. Now let $(x_n^*)_{n=1}^\infty$ be a bounded sequence in $X^*$ which is biorthogonal to $(x_n)_{n=1}^\infty$

and let $(y_n^*)_{n=1}^{\infty}$ be a sequence in $L^{\infty}(\mathbb{T})$ defined by $y_n^*(e^{i\theta}) = e^{-in\theta}$. Clearly, $(x_n^* \otimes y_n^*)_{n=1}^{\infty}$ is a sequence in $(H^p(\mathbb{T}, X))^*$, and for each $n \in \mathbb{N}$, $(x_n^* \otimes y_n^*)(x_n \otimes y_n) = 1$. Also, if $f \in H^p(\mathbb{T}, X)$, then $(x_n^* \otimes y_n^*)(f) = x_n^*(\hat{f}(n))$ and $x_n^*(\hat{f}(n)) \to 0$ as $n \to \infty$, since $\|\hat{f}(n)\| \to 0$ as $n \to \infty$. To see this, define $S_f \colon L^{\infty}(\mathbb{T}) \to X$ by

$$S_f(g) = \int_0^{2\pi} g(e^{i\theta}) f(e^{i\theta}) \frac{d\theta}{2\pi} \quad \text{for } g \in L^{\infty}(\mathbb{T}).$$

$S_f$ is a compact linear operator [3], so $\{\hat{f}(n)\}_{n=1}^{\infty} = \{S_f(e^{-in\theta})\}_{n=1}^{\infty}$ is a relatively compact subset of $X$. If $x^* \in X^*$, then

$$x^*(\hat{f}(n)) = x^* S_f(e^{-in\theta}) = \int_0^{2\pi} x^* f(e^{i\theta}) e^{-in\theta} \frac{d\theta}{2\pi} \to 0$$

as $n \to \infty$ since $x^* f \in L^p(\mathbb{T})$ and the Riemann-Lebesgue lemma. Therefore $(\hat{f}(n))_{n=1}^{\infty}$ converges weakly to 0 and hence converges to 0 in norm.

Thus $(x_n^* \otimes y_n^*)_{n=1}^{\infty}$ is weak* null so $(x_n \otimes y_n)_{n=1}^{\infty}$ and $(x_n^* \otimes y_n^*)_{n=1}^{\infty}$ satisfy the conditions of the lemma, which completes the proof.

*Remark* 1. It is clear that this proof can be used in the following setting: Let $G$ be a compact abelian group with normalized Haar measure on $G$. Let $\hat{G}$ be the dual group of $G$, and let $\Lambda$ be a subset of $\hat{G}$. For $1 \le p \le \infty$ and a complex Banach space $X$, we define

$$L_{\Lambda}^p(G, X) = \{f \in L^p(G, X) \colon \hat{f}(\gamma) = 0 \quad \text{for all } \gamma \notin \Lambda\}.$$

If $X$ contains a copy of $c_0$, if $1 \le p < \infty$, and if $\Lambda$ is infinite, then $L_{\Lambda}^p(G, X)$ contains a complemented copy of $c_0$. Note that if $f \in L^1(G)$, then the net $(\hat{f}(\gamma))_{\gamma \in \Lambda}$ is an element of $c_0(\Lambda)$ (see [6]).

*Remark* 2. The conclusion of the theorem does not hold true if $H^p(\mathbb{T}, X)$ is replaced by $H^p(D, X)$. For example, consider $H^p(D, \ell_{\infty})$ for $1 \le p \le \infty$. By a result of Dowling [4], $H^p(D, \ell_{\infty})$ is a dual Banach space for $1 \le p \le \infty$. However, Bessaga and Pelczynski [1] have proved that $c_0$ is never complemented in the dual of a Banach space. Therefore, $H^p(D, \ell_{\infty})$ does not contain complemented copies of $c_0$. We know that $H^p(\mathbb{T}, \ell_{\infty})$ is isomorphic to a subspace of $H^p(D, \ell_{\infty})$, so the results of this note show that $H^p(\mathbb{T}, \ell_{\infty})$ is not isomorphic to a complemented subspace of $H^p(D, \ell_{\infty})$ when $1 \le p < \infty$.

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