NEW COMBINATORIAL INTERPRETATIONS OF TWO ANALYTIC IDENTITIES

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ABSTRACT. Two generalized partition theorems involving partitions with “ $n + 1$ copies of n ” and “ $n + 2$ copies of n ”, respectively, are proved. These theorems have potential of yielding infinite Rogers-Ramanujan type identities on MacMahon’s lines. Five particular cases are also discussed. Among them three are known and two provide new combinatorial interpretations of two known q -identities.

1. INTRODUCTION, DEFINITIONS, AND MAIN RESULTS

In this paper we shall prove some partition identities involving partitions with “ $n + \ell$ copies on n ”. We first recall the following definitions from [3]:

Definition 1. A partition with “ $n + \ell$ copies of n ”, $\ell \geq 0$ is a partition in which a part of size n , $n \geq 0$, can come in $n + \ell$ different colors denoted by subscripts: $n_1, n_2, \dots, n_{n+\ell}$. In the part n_i , n can be zero if and only if $i \geq 1$. But in no partition are zeros permitted to repeat.

Definition 2. The weighted difference of two parts m_i, n_j , $m \geq n$ is defined by $m - n - i - j$ and is denoted by $((m_i - n_j))$.

Recently in [1] the following result was proved:

Theorem 1. For $k \geq -3$, let $A_k(\nu)$ denote the number of partitions of ν with “ n copies of n ” such that the weighted difference of each pair of summands m_i, n_j is greater than k . Then

$$(1.1) \quad \sum_{\nu=0}^{\infty} A_k(\nu) q^{\nu} = \sum_{\nu=0}^{\infty} \frac{q^{\nu \left[1 + \frac{(k+3)(\nu-1)}{2} \right]}}{(q; q)_{\nu} (q; q^2)_{\nu}},$$

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where

$$(a; q)_n = \prod_{i=0}^{\infty} \frac{(1 - aq^i)}{(1 - aq^{n+i})}.$$

Some particular cases were also discussed. Among them was the following identity in which the partitions enjoy a convolution property:

Corollary 1. $A_{-2}(\nu)$ equals $\sum_{k=0}^{\nu} A_{\nu-k} B_k$, where A_{ν} denotes the number of partitions of ν into distinct parts $\equiv \pm 3 \pmod{7}$ and B_{ν} denotes the number of partitions of ν into parts $\not\equiv 0, \pm 4 \pmod{14}$.

Corollary 1 is a combinatorial interpretation of the identity [5, Eq. (3.1), p. 219]:

$$(1.2) \quad \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}(n^2+n)}}{(q; q)_n (q; q^2)_n} = \prod_{n=1}^{\infty} \frac{(1 + q^n)(1 - q^{7n-2})(1 - q^{7n-5})(1 - q^{7n})}{(1 - q^n)(1 + q^{7n-1})(1 + q^{7n-6})}.$$

In this paper we shall prove the following two theorems, which are very similar to Theorem 1:

Theorem 2. For $k \geq -3$, let $B_k(\nu)$ denote the number of partitions of ν with " $n + 1$ copies of n " such that the weighted difference of each pair of parts is greater than k , the parts are nonnegative, and for some i , i_{i+1} is a part. Then

$$(1.3) \quad \sum_{\nu=0}^{\infty} B_k(\nu) q^{\nu} = \sum_{\nu=0}^{\infty} \frac{q^{\nu(\nu+1)(k+3)/2}}{(q; q)_{\nu} (q; q^2)_{\nu+1}}.$$

Theorem 3. For $k \geq -3$, let $C_k(\nu)$ denote the number of partitions of ν with " $n + 2$ copies of n " into nonnegative parts such that the weighted difference of each pair of parts is greater than k , and for some i , i_{i+2} is a part. Then

$$(1.4) \quad \sum_{\nu=0}^{\infty} C_k(\nu) q^{\nu} = \sum_{\nu=0}^{\infty} \frac{q^{\nu[1+(\nu+1)(k+3)/2]}}{(q; q)_{\nu} (q; q^2)_{\nu+1}}.$$

In §3 we shall show that Theorem 2 and 3 provide new combinatorial interpretations (similar to Corollary 1 above) of the following q -identities [6, p. 160 I(80) and I(82), respectively]:

$$(1.5) \quad \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n+1)}}{(q; q)_n (q; q^2)_{n+1}} = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \prod_{n=1}^{\infty} (1 - q^{7n-2})(1 - q^{14n-11})(1 - q^{14n-3})(1 - q^{7n})$$

and

$$(1.6) \quad \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n+3)}}{(q; q)_n (q; q^2)_{n+1}} = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \prod_{n=1}^{\infty} (1 - q^{7n-3})(1 - q^{7n-4})(1 - q^{14n-13})(1 - q^{14n-1})(1 - q^{7n}).$$

In §3 we shall pose some very significant open problems.

2. PROOFS OF THEOREMS 2 AND 3

Proof of Theorem 2. Let $A_k(m, \nu)$ and $B_k(m, \nu)$ denote respectively the number of partitions of ν enumerated by $A_k(\nu)$ and $B_k(\nu)$ with the added restriction that there be exactly m parts.

It was shown in [1, Eq. (2.6)] that if

$$(2.1) \quad f_k(z, q) = \sum_{m=0}^{\infty} \sum_{\nu=0}^{\infty} A_k(m, \nu) z^m q^{\nu},$$

then

$$(2.2) \quad f_k(z, q) = \sum_{\nu=0}^{\infty} \frac{q^{\nu[1+(k+3)(\nu-1)/2]} z^{\nu}}{(q; q)_{\nu} (q; q^2)_{\nu}};$$

and so

$$(2.3) \quad f_k(z, q) - f_k(zq, q) = zq \sum_{\nu=0}^{\infty} \frac{q^{\nu(\nu+1)(k+3)/2} (zq)^{\nu}}{(q; q)_{\nu} (q; q^2)_{\nu+1}}.$$

Setting

$$(2.4) \quad \begin{aligned} g_k(z, q) &= \sum_{\nu=0}^{\infty} \frac{q^{\nu(\nu+1)(k+3)/2} z^{\nu}}{(q; q)_{\nu} (q; q^2)_{\nu+1}} \\ &= \sum_{m=0}^{\infty} \sum_{\nu=0}^{\infty} E_k(m, \nu) z^m q^{\nu}, \end{aligned}$$

we see by coefficient comparison in (2.3) that

$$(2.5) \quad A_k(m, \nu) - A_k(m, \nu - m) = E_k(m - 1, \nu - m).$$

Equation (2.5) shows that $E_k(m, \nu)$ equals the number of partitions of $\nu + m + 1$ with “ n copies n ” into $m + 1$ parts such that the weighted difference of each pair of parts m_i, n_j is greater than k and, for some i , i_i is a part. If we subtract 1 from each part of a partition enumerated by $E_k(m, \nu)$ ignoring the subscripts we see that the resulting partition is enumerated by $B_k(m + 1, \nu)$. This implies that

$$(2.6) \quad E_k(m, \nu) = B_k(m + 1, \nu).$$

Hence

$$(2.7) \quad \sum_{m=0}^{\infty} \sum_{\nu=0}^{\infty} B_k(m + 1, \nu) z^m q^{\nu} = \sum_{m=0}^{\infty} \sum_{\nu=0}^{\infty} \frac{q^{\nu(\nu+1)(k+3)/2} z^m}{(q; q)_{\nu} (q; q^2)_{\nu+1}}.$$

Now

$$\sum_{\nu=0}^{\infty} B_k(\nu) q^{\nu} = \sum_{\nu=0}^{\infty} \left[\sum_{m=0}^{\infty} B_k(m, \nu) \right] q^{\nu} = g_k(1, q) = \sum_{\nu=0}^{\infty} \frac{q^{\nu(\nu+1)(k+3)/2}}{(q; q)_{\nu} (q; q^2)_{\nu+1}}.$$

This completes the proof of Theorem 2.

Proof of Theorem 3. Let $C_k(m, \nu)$ denote the number of partitions of ν counted by $C_k(\nu)$ with the added restriction that there be exactly m parts. Equation (2.3) can be written as

$$(2.8) \quad f_k(z, q) - f_k(zq, q) = zqh_k(z, q),$$

where

$$(2.9) \quad h_k(z, q) = \sum_{\nu=0}^{\infty} \frac{q^{\nu[1+(\nu+1)(k+3)/2]} z^{\nu}}{(q; q)_{\nu} (q; q^2)_{\nu+1}}.$$

Setting

$$(2.10) \quad h_k(z, q) = \sum_{m=0}^{\infty} \sum_{\nu=0}^{\infty} F_k(m, \nu) z^m q^{\nu},$$

we see by coefficient comparison in (2.8) that

$$(2.11) \quad A_k(m, \nu) - A_k(m, \nu - m) = F_k(m - 1, \nu - 1).$$

Equation (2.11) shows that $F_k(m, \nu)$ equals the number of partitions of $\nu + 1$ with “ n copies of n ” into $m + 1$ parts such that the weighted difference of each pair of parts m_i, n_j is greater than k and for some i, i_i is a part. If we replace this part i_i by $(i - 1)_{i+1}$, we see that the resulting partition is enumerated by $C_k(m + 1, \nu)$. This implies that

$$(2.12) \quad F_k(m, \nu) = C_k(m + 1, \nu).$$

Hence

$$\sum_{m=0}^{\infty} \sum_{\nu=0}^{\infty} C_k(m + 1, \nu) z^m q^{\nu} = \sum_{m=0}^{\infty} \sum_{\nu=0}^{\infty} \frac{q^{\nu[1+(\nu+1)(k+3)/2]} z^{\nu}}{(q; q)_{\nu} (q; q^2)_{\nu+1}}.$$

Now

$$\begin{aligned} \sum_{\nu=0}^{\infty} c_k(\nu) q^{\nu} &= \sum_{\nu=0}^{\infty} \left\{ \sum_{m=0}^{\infty} C_k(m, \nu) \right\} q^{\nu} = h_k(1, q) \\ &= \sum_{\nu=0}^{\infty} \frac{q^{\nu[1+(\nu+1)(k+3)/2]}}{(q; q)_{\nu} (q; q^2)_{\nu+1}}, \end{aligned}$$

and the proof of Theorem 3 is completed.

3. PARTICULAR CASES

For $k = 0$, Theorem 2, in view of the identity [6, I(4), p. 156]

$$(3.1) \quad \sum_{n=0}^{\infty} \frac{q^{3n(n+1)/2}}{(q; q)_n (q; q^2)_n} = \frac{1}{(q; q)_{\infty}} \prod_{n=1}^{\infty} (1 - q^{10n})(1 - q^{10n-2})(1 - q^{10n-8}),$$

yields

Corollary 2.1. *The number of partitions of ν with “ $n + 1$ copies of n ” into nonnegative parts, such that each pair of parts m_i, n_j has positive weighted difference and for some i , i_{i+1} is a part, equals the number of ordinary partitions of ν into parts $\not\equiv 0, \pm 2 \pmod{10}$.*

For $k = -1$, Theorem 2, in view of the identity [6, I(60), p. 158]

$$(3.2) \quad \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q; q)_n (q; q^2)_{n+1}} = \frac{1}{(q; q)_{\infty}} \prod_{n=1}^{\infty} (1 - q^{14n-4})(1 - q^{14n-10})(1 - q^{14n}),$$

reduces to

Corollary 2.2. *The number of partitions of ν with “ $n + 1$ copies of n ” into nonnegative parts, such that each pair of parts m_i, n_j has nonnegative weighted difference and for some i , i_{i+1} is a part, equals the number of ordinary partitions of ν into parts $\not\equiv 0, \pm 4 \pmod{14}$.*

On the other hand, in this particular case, Theorem 3, in view of the identity [6, I(59), p. 157]

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q; q)_n (q; q^2)_{n+1}} = \frac{1}{(q; q)_{\infty}} \prod_{n=1}^{\infty} (1 - q^{14n-2})(1 - q^{14n-12})(1 - q^{14n}),$$

reduces to

Corollary 3.1. *The number of partitions of ν with “ $n + 2$ copies of n ” into nonnegative parts, such that each pair of parts m_i, n_j has nonnegative weighted difference and for some i , i_{i+2} is a part, equals the number of ordinary partitions of ν into parts $\not\equiv 0, \pm 2 \pmod{14}$.*

Remark 1. Corollaries 2.1, 2.2, and 3.1 are the particular cases $k = 2, \ell = 1$; $k = 3, \ell = 1$; and $k = 3, \ell = 2$, respectively of Theorem 4 in [3].

Remark 2. Another proof of Corollaries 2.1, 2.2, and 3.1 can be found in [4]. For $k = -2$, Theorems 2 and 3 give the following combinatorial interpretations, presumably new, of the identities (1.5) and (1.6), respectively.

Corollary 2.3. *The number of partitions of ν with “ $n + 1$ copies of n ” into nonnegative parts, such that the weighted difference of each pair of summands m_i, n_j is greater than or equal to -1 and for some i , i_{i+1} is a part, equals $\sum_{k=0}^{\nu} C_k D_{\nu-k}$, where C_k denotes the number of partitions k into parts $\equiv \pm 1, \pm 4, \pm 6 \pmod{14}$ and D_k denotes the number of partitions of k into distinct parts.*

Example. $B_{-2}(5) = 12$, since the relevant partitions are $5_6, 5_1 + 0_1, 5_2 + 0_1, 5_3 + 0_1, 5_4 + 0_1, 5_5 + 0_1, 4_1 + 1_2, 4_2 + 1_2, 4_1 + 1_1 + 0_1, 4_2 + 1_1 + 0_1, 4_3 + 1_1 + 0_1, 3_1 + 2_1 + 0_1$.

Also,

$$\begin{aligned}\sum_{k=0}^5 C_k D_{5-k} &= C_0 D_5 + C_1 D_4 + C_2 D_3 + C_3 D_2 + C_4 D_1 + C_5 D_0 \\ &= 1(3) + 1(2) + 1(2) + 1(1) + 2(1) + 2(1) \\ &= 12.\end{aligned}$$

Corollary 3.2. *The number of partitions of ν with “ $n + 2$ copies of n ” into nonnegative parts, such that the weighted difference of each pair of summands m_i, n_j is greater than or equal to -1 and for some i, i_{i+2} is a part, equals $\sum_{k=0}^{\nu} E_k D_{\nu-k}$, where E_k denotes the number of partitions of k into parts $\equiv 2, 5, 6, 8, 9, 12 \pmod{14}$ and D_k , as in Corollary 2.3, denotes the number of partitions of k into distinct parts.*

Example. $C_{-2}(6) = 10$, since the relevant partitions are $6_8, 6_1 + 0_2, 6_2 + 0_2, 6_3 + 0_2, 6_4 + 0_2, 6_5 + 0_2, 4_1 + 2_1 + 0_2, 4_2 + 2_1 + 0_2, 5_1 + 1_3, 5_2 + 1_3$. Also,

$$\begin{aligned}\sum_{\nu=0}^6 E_k D_{6-k} &= E_0 D_6 + E_1 D_5 + E_2 D_4 + E_3 D_3 + E_4 D_2 + E_5 D_1 + E_6 D_0 \\ &= 1(4) + 0(3) + 1(2) + 0(2) + 1(1) + 1(1) + 2(1) \\ &= 10.\end{aligned}$$

4. CONCLUSION

Many questions arise from this work. The most obvious among them are:

- (1) Can Theorems 1, 2, and 3 be combined into one?
- (2) Is it possible to give a nice combinatorial interpretation of Theorems 1, 2, and 3 for the general value of k ?
- (3) The methods used here were also used recently in [2] to give n -color partitions theoretic interpretations of several q -identities from [6]. Is it possible to prove Theorems 3 and 4 of [3] by using these methods?
- (4) We have seen that each of Theorems 1, 2, and 3 yields a partition identity wherein the partitions enjoy a convolution property. Does this lead to the conclusion that there exists an infinite family of identities wherein partitions enjoy a convolution property?

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