UNITARY REPRESENTATIONS OF LIE GROUPS AND GÄRDING'S INEQUALITY

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ABSTRACT. We prove two versions of Gårding's inequality for strongly elliptic operators in the enveloping Lie algebra associated with a unitary representation of a Lie group. We then deduce a characterization of the differential structure of the representation in terms of the elliptic operators.

1. INTRODUCTION

Let $(\mathcal{H}, G, U)$ denote a continuous representation of the connected Lie group G by unitary operators $U(g)$, $g \in G$, on the Hilbert space $\mathcal{H}$. Fix a basis $a_1, \dots, a_d$ of the Lie algebra $\mathfrak{g}$ of G and let $A_1, \dots, A_d$ denote the skew self-adjoint generators of the one-parameter subgroups $t \in \mathbb{R} \mapsto U(e^{-ta_i})$. If $\alpha = (\alpha_1, \dots, \alpha_d)$, with $\alpha_i \geq 0$ and $|\alpha| = \alpha_1 + \dots + \alpha_d$, we define $A^\alpha = A_1^{\alpha_1} A_2^{\alpha_2} \dots A_d^{\alpha_d}$ and set $\mathcal{H}_\infty = \bigcap_\alpha D(A^\alpha)$. It follows by standard reasoning that $\mathcal{H}_\infty$ is norm dense in $\mathcal{H}$.

A form

$$\xi \in \mathbb{R}^d \rightarrow C_m(\xi) = \sum_{\alpha; |\alpha| \leq m} c_\alpha \xi^\alpha \in \mathbb{C},$$

where $\xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \dots \xi_d^{\alpha_d}$, is defined to be *strongly elliptic* if $\operatorname{Re}((-1)^{m/2} P_m(\xi)) > 0$ for all $\xi \in \mathbb{R}^d \setminus \{0\}$, where P_m denotes the *principal part* of C_m , i.e.,

$$P_m = \sum_{\alpha; |\alpha|=m} c_\alpha \xi^\alpha.$$

Equivalently, C_m is strongly elliptic if there is a $p > 0$ such that

$$\operatorname{Re}((-1)^{m/2} P_m(\xi)) \geq p|\xi|^m$$

for all $\xi \in \mathbb{R}^d$. The largest value $p_m (= p_m(c))$ of p for which this estimate is valid is called the *ellipticity constant* of C_m . Note that the strong ellipticity condition implies automatically that m is even.

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Next we associate with each unitary representation $(\mathcal{H}, G, U)$, each basis $a_1, \dots, a_d$, of $\mathfrak{g}$, and each strongly elliptic form C_m , an operator

$$A_m(c) = \sum_{\alpha; |\alpha| \leq m} c_\alpha A^\alpha$$

with domain $D(A_m) = \mathcal{H}_\infty$. We refer to the A_m as strongly elliptic operators. The simplest example is the Laplacian

$$\Delta = - \sum_{i=1}^d A_i^2$$

corresponding to the form $\xi \mapsto -|\xi|^2$.

Theorem 1.1. *Let $A_m(c)$ be strongly elliptic. For each $p \in \langle 0, p_m(c) \rangle$ there is a $q \geq 0$ such that*

$$(1.1) \quad \operatorname{Re}(x, A_m(c)x) \geq p(x, \Delta^{m/2}x) - q(x, x)$$

for all $x \in \mathcal{H}_\infty$. Moreover, q can be chosen independently of the particular unitary representation.

An alternative, weaker version of the theorem can be stated in terms of the C^n -norms of the representation. These are defined by $\|x\|_0 = \|x\|$ and

$$\|x\|_n = \sup_{0 \leq i \leq d} \|A_i x\|_{n-1},$$

where $A_0 = I$. The C^n -subspace

$$\mathcal{H}_n = \bigcap_{\alpha; |\alpha| \leq n} D(A^\alpha)$$

is a Banach space with respect to $\|\cdot\|_n$, and $\mathcal{H}_\infty$ is $\|\cdot\|_n$ -dense in $\mathcal{H}_n$ (see, for example, [G]).

Theorem 1.2. *Let $A_m(c)$ be strongly elliptic. For each $p' \in \langle 0, p_m(c) \rangle$ there is a $q' \geq 0$ such that*

$$(1.2) \quad \operatorname{Re}(x, A_m(c)x) \geq p' \|x\|_{m/2}^2 - q' \|x\|^2$$

for all $x \in \mathcal{H}_\infty$. Moreover, q can be chosen independently of the particular unitary representation.

It was established by R. Goodman [G] that $\mathcal{H}_n = D(\Delta^{n/2})$ and the norm $\|\cdot\|_n$ on $\mathcal{H}_n$ is equivalent to the graph norm $x \mapsto \|\Delta^{n/2}x\| + \|x\|$. But more recently ([R1], [R2]) the following more precise estimates have been obtained; for each $n = 1, 2, \dots$ and $\epsilon > 0$ there is a $c_n(\epsilon)$ such that

$$\|x\|_n \leq (1 + \epsilon) \|\Delta^{n/2}x\| + c_n(\epsilon) \|x\|$$

for all $x \in \mathcal{H}_n$. Therefore Theorem 1.2 follows easily from Theorem 1.1. Next we sketch the proof of the latter result. It is essentially a consequence of the work of Langlands [L1], [L2].

2. OUTLINE OF THE PROOF OF THEOREM 1.1

If C_m is a strongly elliptic form with ellipticity constant p_m and $p \in \langle 0, p_m \rangle$ then the form C'_m defined by

$$C'_m(\xi) = C_m(\xi) - p(-|\xi|^2)^{m/2}$$

is also strongly elliptic, with ellipticity constant $p'_m = p_m - p$. Since $A_m(c') = A_m(c) - p\Delta^{m/2}$ the inequality (1.1) is equivalent to the lower semiboundedness property

$$\operatorname{Re}(x, A_m(c')x) \geq -q(x, x).$$

Therefore to prove Theorem 1.1 it suffices to prove that the real part of every strongly elliptic operator $A_m(c)$ is lower semibounded.

Let $A_m^\dagger$ denote the formal adjoint of A_m on $\mathcal{H}_\infty$, i.e.,

$$A_m^\dagger = \sum_{\alpha; |\alpha| \leq m} \bar{c}_\alpha (-1)^{|\alpha|} A^{\alpha*}$$

on $\mathcal{H}_\infty$ where $A^{\alpha*} = A_d^{\alpha_d} \cdots A_1^{\alpha_1}$. Then $R_m = (A_m + A_m^\dagger)/2$ is a symmetric operator on $\mathcal{H}_\infty$. But it follows from the structure relations of $\mathfrak{g}$ that $A^{\alpha*} = A^\alpha$ modulo lower order terms, i.e.,

$$A^{\alpha*} = A^\alpha + \sum_{\beta; |\beta| < |\alpha|} c_{\alpha, \beta} A^\beta,$$

where the $c_{\alpha, \beta}$ are polynomials in the structure constants. Therefore R_m is a strongly elliptic operator associated with a form C'_m whose principal part P'_m is given by

$$P'_m(\xi) = \sum_{\alpha; |\alpha| = m} (\operatorname{Re} c_\alpha) \xi^\alpha.$$

Now it follows from Langlands' [L1] first theorem that R_m is essentially self-adjoint, and from his second theorem that the self-adjoint closure $\bar{R}_m$ generates a continuous semigroup, which is automatically self-adjoint. But then $\bar{R}_m$ is lower semibounded by spectral theory. Hence there is a $q \geq 0$ such that

$$\operatorname{Re}(x, A_m x) = (x, R_m x) \geq -q(x, x)$$

for all $x \in \mathcal{H}_\infty$. Therefore (1.1) follows from the previous reasoning.

Finally, Langlands' [L1] third theorem establishes that the semigroup S generated by $\bar{R}_m$ has a representation independent kernel, i.e.,

$$S_t = \int_G dg p_t(g) U(g),$$

where dg denotes the left invariant Haar measure and $p_t \in L_1(G; dg)$. Since

$$e^{qt} \leq \int_G dg |p_t(g)|$$

one can then choose q to be independent of the particular unitary representation. This completes the outline of the proof.

3. DIFFERENTIAL STRUCTURE

Let $A_m = A_m(c)$ be a strongly elliptic operator with formal adjoint $A_m^\dagger$ and define

$$B_{2m} = A_m^\dagger A_m = \sum_{\alpha; |\alpha| \leq m} \sum_{\beta; |\beta| \leq m} \bar{c}_\alpha c_\beta (-1)^{|\alpha|} A^{\alpha*} A^\beta$$

on $\mathcal{H}_\infty$. Since $A^{\alpha*} A^\beta = A^{\alpha+\beta}$ modulo lower-order terms, it follows that B_{2m} is a strongly elliptic operator. Moreover, if P_m denotes the principal part of the form C_m associated with A_m and P'_{2m} the principal part of the form C'_{2m} associated with B_{2m} then

$$P'_{2m}(\xi) = |P_m(\xi)|^2 \geq (\operatorname{Re} P_m(\xi))^2.$$

Therefore one has the inequality $p'_{2m} \geq p_m^2$ for the ellipticity constants, with equality whenever the principal part of C_m is real. Now applying Theorem 1.2 to B_{2m} one deduces the following.

Corollary 3.1. *Let $A_m(c)$ be strongly elliptic. For each $p \in \langle 0, p_m(c) \rangle$ there is a $q \geq 0$, independent of the representation, such that*

$$(3.1) \quad \|x\|_m \leq (1/p) \|A_m(c)x\| + q \|x\|$$

for all $x \in \mathcal{H}_\infty$. Consequently $A_m(c)$ is closed on $\mathcal{H}_m$ and the C^m -norm $\|\cdot\|_m$ is equivalent to the graph $x \mapsto \|A_m(c)x\| + \|x\|$.

Proof. Replacing $A_m(c)$ by $B_{2m} = A_m(c)^\dagger A_m(c)$ in (1.2) one finds for each $p' \in \langle 0, p'_{2m} \rangle$ a $q' \geq 0$ such that

$$p' \|x\|_m^2 \leq \|A_m(c)x\|^2 + q' \|x\|^2$$

for all $x \in \mathcal{H}_\infty$. But $p'_{2m} \geq p_m^2$. Thus if $p \in \langle 0, p_m \rangle$ and $p' = p^2$,

$$\|x\|_m^2 \leq (1/p^2) \|A_m(c)x\|^2 + q' \|x\|^2.$$

Then (3.1) follows by elementary reasoning. But

$$(3.2) \quad \|A_m(c)x\| \leq \left[\sum_{\alpha; |\alpha| \leq m} |c_\alpha| \right] \|x\|_m$$

and since $\mathcal{H}_\infty$ is $\|\cdot\|_m$ -dense in $\mathcal{H}_m$ one immediately deduces the last statement of the corollary from (3.1) and (3.2).

Finally the foregoing reasoning extends to higher-order products. If $A_{m_1}, \dots, A_{m_n}$ are all strongly elliptic and $m = m_1 + \dots + m_n$ then

$$B_{2m} = (A_{m_n}^\dagger \cdots A_{m_1}^\dagger)(A_{m_1} \cdots A_{m_n})$$

is a strongly elliptic operator of order $2m$. Hence the same arguments show that the C^m -norm is equivalent to the graph norm

$$x \mapsto \|A_{m_1} \cdots A_{m_n} x\| + \|x\|.$$

Detailed proofs of these results will appear in [R2].

4. SEMIGROUPS BOUNDS

The closure $\overline{A}_m$ of each strongly elliptic operator A_m generates a strongly continuous semigroup S holomorphic in a sector $\Delta_m(\varphi) = \{z \in \mathbb{C}; \operatorname{Re} z > 0, |\operatorname{Arg} z| < \varphi\}$ by Langlands' second theorem. Then if $\theta \in [0, \varphi)$ it follows by general theory that there exist $M_\theta \geq 1$ and $\omega_\theta \geq 0$ such that

$$\|S_z\| \leq M_\theta e^{\omega_\theta |z|}$$

whenever $\operatorname{Re} z \geq 0$ and $|\operatorname{Arg} z| \leq \theta$. But the Gårding inequalities allow one to infer that $M_\theta = 1$, at least for small θ .

Corollary 4.1. *Let C_m be a strongly elliptic form with ellipticity constant p_m , define*

$$q_m = \sum_{\alpha; |\alpha|=m} |\operatorname{Im} c_\alpha|,$$

and $\varphi_m = \tan^{-1} p_m/q_m$. Further let S denote the holomorphic semigroup generated by $\overline{A_m(c)}$.

If $\theta \in [0, \varphi_m)$ then there is an $\omega_\theta \geq 0$ such that

$$\|S_z\| \leq e^{\omega_\theta |z|}$$

for all $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$, and $|\operatorname{Arg} z| \leq \theta$.

Proof. First, by Langlands' estimates [L2], or by [R2], the semigroup S is holomorphic in the sector $\Delta_m(\varphi_m)$, and possibly in a larger sector. Thus if $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$ and $|\operatorname{Arg} z| \leq \theta$ then zA_m generates a continuous semigroup. But

$$\begin{aligned} \operatorname{Re}(x, zA_m x) &= (\operatorname{Re} z)(\operatorname{Re}(x, A_m x) - (\operatorname{Im} z)\operatorname{Im}(x, A_m x)) \\ &\geq (\operatorname{Re} z) \operatorname{Re}(x, A_m x) - |\operatorname{Im} z| q_m \|x\|_{m/2}^2 \\ &\quad - |\operatorname{Im} z| r_m \|x\|_{m/2} \cdot \|x\|_{m/2-1}, \end{aligned}$$

where

$$r_m = \sum_{\alpha; |\alpha| \leq m} |\operatorname{Im} c_\alpha|.$$

Now for each $\delta > 0$,

$$\|x\|_{m/2} \cdot \|x\|_{m/2-1} \leq \delta \|x\|_{m/2}^2 + (1/4\delta) \|x\|_{m/2-1}^2.$$

Moreover, for each $\sigma > 0$ there is a $k_\sigma > 0$ such that

$$\|x\|_{m/2-1}^2 \leq \sigma \|x\|_{m/2}^2 + k_\sigma \|x\|^2.$$

Hence for each $\varepsilon > 0$ there is a $c_\varepsilon > 0$ such that

$$\operatorname{Re}(x, zA_mx) = (\operatorname{Re} z) \operatorname{Re}(x, A_mx) - |\operatorname{Im} z|(q_m + \varepsilon)\|x\|_{m/2}^2 - |\operatorname{Im} z|c_\varepsilon\|x\|^2.$$

Now we can use the second form of Gårding's inequality (1.2) to deduce that for each $p' \in \langle 0, p_m \rangle$ there is a $q' > 0$ such that

$$\operatorname{Re}(x, zA_mx) \geq (\operatorname{Re} z) \operatorname{Re}(x, A_mx) \left(1 - \frac{|\operatorname{Im} z|}{\operatorname{Re} z} \frac{q_m + \varepsilon}{p'}\right) - |\operatorname{Im} z| \cdot \|x\|^2 \left(q_m \frac{q'}{p'} + c_\varepsilon\right).$$

But by choosing p' close to p_m and ε small, one can assure that $(1 - (q_m + \varepsilon)|\operatorname{Im} z|/p' \operatorname{Re} z) \geq 0$. Then by another application of Gårding's inequality there is a $q \geq 0$ such that $\operatorname{Re}(x, A_mx) \geq -q\|x\|^2$. Therefore

$$\operatorname{Re}(x, (z/|z|)A_mx) \geq -\omega_\theta\|x\|^2$$

with $\omega_\theta = q(1 - ((q_m + \varepsilon)/p') \tan \theta) + q'q_m/p' + c_\varepsilon$.

Finally,

$$\frac{d}{d|z|} \|S_z x\|^2 e^{-2\omega_\theta|z|} = -\operatorname{Re}(S_z x, ((z/|z|)A_m + \omega_\theta I)S_z x) e^{-2\omega_\theta|z|} \leq 0.$$

Therefore, by integration,

$$\|S_z x\| \leq e^{\omega_\theta|z|} \|x\|$$

for all $z \in \mathbb{C}$ with $\operatorname{Re} z \geq 0$ and $|\operatorname{Arg} z| \leq \theta$.

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