THE TORUS LEMMA ON CALIBRATIONS, EXTENDED

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ABSTRACT. The whole face $G(\varphi)$ of m -planes calibrated by a torus m -form φ is determined by the torus face $G_T(\varphi)$. Indeed, $G(\varphi)$ results from applying a new closure operation to $G_T(\varphi)$.

1. INTRODUCTION

Over the past ten years the theory of calibrations has illuminated the occurrence and structure of singularities in m -dimensional area-minimizing surfaces. This note gives an extension of a much-used lemma on calibrations, the Torus Lemma (cf. §3). Our observations bear on recent work of D. Nance [N, e.g. Corollary 3.8] and M. Messaoudene [Me].

For surveys on calibrations see [H1], [M1], [M2]. For basic concepts and definitions see [M3, §1, §2], [M4, Chapter 4], the original paper [HL], or the new text [H2].

2. DEFINITIONS

In addition to the standard dual Euclidean norms on the exterior algebra $\bigwedge^m \mathbf{R}^n$ and its dual $\bigwedge^m \mathbf{R}^{n*}$, there is another important dual pair of norms, called mass and comass. The *comass* $\|\varphi\|^*$ of a form $\varphi \in \bigwedge^m \mathbf{R}^{n*}$ is the maximum value of φ on the Grassmannian $G(m, \mathbf{R}^n)$ of oriented unit m -planes through 0 in $\mathbf{R}^n$:

$$\|\varphi\|^* = \max\{\langle \xi, \varphi \rangle : \xi \in G(m, \mathbf{R}^n)\}.$$

A form φ normalized to have comass 1 is called a *calibration*. The *face* $G(\varphi)$ of a calibration φ consists of its maximum points in the Grassmannian:

$$G(\varphi) = \{\xi \in G(m, \mathbf{R}^n) : \langle \xi, \varphi \rangle = 1\}.$$

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The mass is the dual norm on m -vectors ξ in $\bigwedge^m \mathbf{R}^n$:

$$\begin{aligned}\|\xi\| &= \max\{\langle \xi, \varphi \rangle : \|\varphi\|^* = 1\} \\ &= \min \left\{ \sum a_j : \xi = \sum a_j \xi_j, \xi_j \in G(m, \mathbf{R}^n), a_j > 0 \right\}.\end{aligned}$$

A calibration φ for which the maximum of $\langle \xi, \varphi \rangle$ is attained is said to *calibrate* ξ . An expression $\xi = \sum a_j \xi_j$ for which the minimum is attained is called a *mass decomposition* for ξ . A calibration φ calibrates $\xi = \sum a_j \xi_j$ if and only if all the ξ_j lie in the face $G(\varphi)$.

In the case that $n = 2m$ and $\mathbf{R}^n = (\mathbf{R}^2)^m$ consider the m -dimensional torus

$$T = (S^1)^m = (G(1, \mathbf{R}^2))^m \subset G(m, \mathbf{R}^{2m}) \subset \bigwedge^m \mathbf{R}^{2m}.$$

The elements of T are called *torus planes*. Let T_s denote the span of T in $\bigwedge^m \mathbf{R}^{2m}$. The *torus span* T_s can also be described as the tensor product $T_s = \bigotimes_{i=1}^m \bigwedge^1 \mathbf{R}^2$. Elements of T_s are called *torus m -vectors*. Similarly the elements of the dualspace T_s^* are called *torus forms*. The intersection of the face $G(\varphi)$ of any calibration φ with the torus T is called the *torus face* $G_T(\varphi)$.

3. THE TORUS LEMMA

The Torus Lemma ([M5, Lemma 4], cf. [DHM, §4]) says that a torus calibration attains its maximum value of 1 on the torus. Equivalently,

$$(*) \quad \left\{ \varphi \in T_s^* : \max_{\xi \in T} \langle \xi, \varphi \rangle = 1 \right\} = \{ \varphi \in T_s^* : \|\varphi\|^* = 1 \}.$$

We observe a few useful consequences.

- (1) The unit mass ball intersects the torus span in the convex hull of T .
- (2) A torus m -vector is calibrated by a torus calibration [Me, 4.4.2].
- (3) A torus m -vector has a mass decomposition in terms of torus m -planes. (This generalizes [Me, 5.2.8, 5.4.3].)
- (4) Let φ be a torus calibration. An m -plane ξ belongs to $G(\varphi)$ if and only if its projection $P\xi$ onto the torus span T_s is a convex combination of $G_T(\varphi)$.

Consequence (4) includes the new observation that *the face $G(\varphi)$ of a torus form φ is determined by the torus face $G_T(\varphi)$* . This fact is applied by D. Nance [N, e.g. Corollary 3.8]. Theorem 6 will exhibit this fact in another way.

The consequences follow immediately from (*). For example, we will verify (4). Since φ is a torus calibration, $\varphi(\xi) = \varphi(P\xi)$. Clearly if $P\xi$ is a convex combination of $G_T(\varphi)$, then $\varphi(\xi) = \varphi(P\xi) = 1$, so that $\xi \in G(\varphi)$. On the other hand, suppose $\xi \in G(\varphi)$, so that $\varphi(P\xi) = \varphi(\xi) = 1$. For any other torus calibration φ' , $\varphi'(P\xi) = \varphi'(\xi) \leq 1$. It follows by elementary convex geometry from the characterization (*) of torus calibrations as

$$\left\{ \varphi \in T_s^* : \max_{\xi \in T} \langle \xi, \varphi \rangle = 1 \right\}$$

that $P\xi$ is a convex combination of $G_T(\varphi)$.

4. DEFINITIONS

We call $A \subset G(m, \mathbf{R}^n)$ a $\mathbf{CP}^1$ if for some orthonormal basis $e_1, \dots, e_n$ for $\mathbf{R}^n$ and the complex structure $Je_1 = e_3, Je_2 = e_4$ on $\mathbf{R}^4 = \text{span}\{e_1, e_2, e_3, e_4\}$.

$$A = \{\text{the complex lines in } \mathbf{R}^4\} \wedge e_5 \wedge \dots \wedge e_{m+2}.$$

Let $B \subset G(m, \mathbf{R}^n)$. We define the $\mathbf{CP}^1$ -closure $C(B)$ of B as the smallest subset of $G(m, \mathbf{R}^n)$ containing B such that whenever two points of a $\mathbf{CP}^1$ lie in $C(B)$, the whole $\mathbf{CP}^1$ lies in $C(B)$.

Proposition 5 ([HM, Corollary 4.7]). *Let $G(\varphi)$ be a face of $G(m, \mathbf{R}^n)$. If two points of a $\mathbf{CP}^1$ lie in $G(\varphi)$, then the whole $\mathbf{CP}^1$ lies in $G(\varphi)$.*

Theorem 6. *Let $\varphi \in \bigwedge^m \mathbf{R}^{2m*}$ be a torus calibration. Then the face $G(\varphi)$ is the $\mathbf{CP}^1$ -closure of the torus face $G_T(\varphi)$:*

$$G(\varphi) = C(G_T(\varphi)).$$

Remarks. This theorem subsumes both the Torus Lemma, which just says that $G_T(\varphi) \neq \emptyset$, and our new observation that the entire face of a torus calibration is determined by its torus face (cf. §3).

Proof. By Proposition 5, $G(\varphi) \supset C(G_T(\varphi))$. We prove the opposite inclusion by induction on m . The result is trivial for $m = 1$. Suppose $\varphi \in \bigotimes_{j=1}^{m+1} (\bigwedge^1 \mathbf{R}^{2*}) \subset (\bigwedge^1 \mathbf{R}^{2*}) \otimes (\bigwedge^m \mathbf{R}^{2m*})$. Let $\xi \in G(\varphi)$. It is not hard to show ([HL, Lemma II.7.5]) that there are orthonormal bases e_1, e_2 for $\mathbf{R}^2$ and $f_1, \dots, f_{2m}$ for $\mathbf{R}^{2m}$ and angles $\theta_1, \theta_2 \in [0, \pi/2]$ such that ξ takes the form

$$\xi = (\cos \theta_1 e_1 + \sin \theta_1 f_1) \wedge (\cos \theta_2 e_2 + \sin \theta_2 f_2) \wedge f_3 \wedge \dots \wedge f_{m+1}.$$

Since $\varphi \in \bigwedge^1 \mathbf{R}^{2*} \otimes \bigwedge^m \mathbf{R}^{2m*}$,

$$\begin{aligned} \varphi(\xi) &= a \cos \theta_1 \sin \theta_2 + b \sin \theta_1 \cos \theta_2 \\ &\leq \sqrt{a^2 \cos^2 \theta_1 + b^2 \sin^2 \theta_1} \leq \max\{|a|, |b|\} \leq 1, \end{aligned}$$

where

$$\begin{aligned} a &= \langle e_1 \wedge f_2 \wedge \dots \wedge f_{m+1}, \varphi \rangle, \\ b &= \langle f_1 \wedge e_2 \wedge \dots \wedge f_{m+1}, \varphi \rangle. \end{aligned}$$

Hence, equality holds. Unless $a = b = 1$, it follows that $\{\theta_1, \theta_2\} = \{0, \pi/2\}$ and ξ has a factor e_1 or e_2 , say e_1 . Thus $\xi = e_1 \wedge \zeta$, for some $\zeta \in G(m, \mathbf{R}^{2m})$. Since $e_1 \lrcorner \varphi \in \bigotimes_{j=1}^m \bigwedge^1 \mathbf{R}^{2*}$ and $\langle \zeta, e_1 \lrcorner \varphi \rangle = \pm \langle e_1 \wedge \zeta, \varphi \rangle = \pm 1$, by induction $\pm \zeta$ lies in $C(G_T(e_1 \lrcorner \varphi))$. Consequently $\xi = e_1 \wedge \zeta$ belongs to

$$\pm e_1 \wedge C(G_T(e_1 \lrcorner \varphi)) \subset \pm C(e_1 \wedge G_T(e_1 \lrcorner \varphi)) \subset C(G_T(\varphi))$$

as desired.

If on the other hand $a = b = 1$, then $\theta_2 = \pi/2 - \theta_1$. Also, $e_1 \wedge f_2 \wedge \cdots \wedge f_{m+1}$ and $f_1 \wedge e_2 \wedge \cdots \wedge f_{m+1}$ both belong to $G(\varphi)$. As in the previous case, by induction both belong to $C(G_T(\varphi))$. In addition for the complex structure $Je_1 = f_2$, $Jf_1 = e_2$, both belong to the $\mathbb{C}P^1$

$$A = \{\text{complex lines in span } \{e_1, f_2, f_1, e_2\}\} \wedge f_3 \wedge \cdots \wedge f_{m+1}.$$

Since $\theta_2 = \pi/2 - \theta_1$, ξ also belongs to A . Therefore $\xi \in C(G_T(\varphi))$, as desired.

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