THE STIELTJES MOMENTS PROBLEM FOR RAPIDLY DECREASING FUNCTIONS

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ABSTRACT. We prove the following result: If $(a_n)_n$ is a sequence of complex numbers, then there exists a $\mathcal{E}^\infty$ -function f such that f and all its derivatives are rapidly decreasing functions, $f(t) = 0$ for $t < 0$ and $\int_0^{+\infty} t^n f(t) dt = a_n$. We extend this result for a generalized Stieltjes moments problem. Also, we characterize the C^∞ -functions f in $(0, +\infty)$ such that f and all its derivatives are rapidly decreasing functions in $(0, +\infty)$ and with null moments.

INTRODUCTION

The problem of determining necessary and sufficient conditions for a sequence of numbers $(a_n)_n$ to have the form $a_n = \int_0^\infty t^n d\alpha(t)$ where $\alpha(t)$ is non-decreasing and $n \geq 0$ was set and solved by T. J. Stieltjes in 1894–95 [12].

In 1938 R. P. Boas and G. Polya [2, 9] found for each sequence $(a_n)_n$ a function f of bounded variation in $(0, +\infty)$ such that $a_n = \int_0^\infty t^n df(t)$.

We prove here that we can find a regular function f with the given moments. Indeed, we prove the following result:

Theorem. *Let $(a_n)_n$ be a sequence of complex numbers; then there exists a function f such that:*

- (a) $f \in S$, where S is the Schwartz's space (i.e. f is a C^∞ -function and f and all its derivatives are rapidly decreasing functions) and $f(t) = 0$ if $t < 0$.
- (b) $a_n = \int_0^\infty t^n f(t) dt$ for all $n \geq 0$.

(If a function f satisfies condition (a), we say that $f \in S^+ \cap S$).

The key in the proof is the characterization of a certain space of analytic functions in the disc $\Omega_0 = \{u \in \mathbb{C}: |u - \frac{1}{2}| < \frac{1}{2}\}$ (this space is obtained from

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$S^+ \cap S$ by taking Fourier transform in a suitable sense) in which an interpolation theorem is proved (see §2).

This characterization is obtained from the results in [3], where we study the Fourier coefficients of functions in the space $S^+ \cap S$ with respect to a certain orthonormal system. We relate the coefficients of a function with its Fourier transform (see §1).

In §4, using certain properties obtained in [3] for a Hankel transform (which was used by Hardy [8] to study a Stieltjes moments problem) defined by

$$\mathcal{H}_0(f)(t) = \frac{1}{2} \int_0^\infty f(x) J_0(\sqrt{xt}) dx,$$

where J_0 is the Bessel function of first kind and f is a suitable function, we characterize in a unexpected way (because of its simplicity), the functions f defined on $(0, +\infty)$, such that f is a C^∞ -function and f and all its derivatives are rapidly decreasing in $(0, +\infty)$ (we shall say that $f \in S^+$) with null moments. Indeed, we prove:

Theorem. *Let ϕ be a C^∞ -function in $(0, +\infty)$ such that $\sup_{t>0} t^k |\phi^{(n)}(t)| < \infty$ for all $k, n \geq 0$. Then $\int_0^\infty t^n \phi(t) dt = 0$ if and only if there exists a function $\psi \in S^+ \cap S$ such that $\phi(t) = \frac{1}{2} \int_0^\infty \psi(x) J_0(\sqrt{xt}) dx$.*

Also, in §4 we propose a problem on the functions of $S^+ \cap S$ with null moments. Finally, we extend the previous results to a generalized moment problem.

Terminology. Throughout the paper S denotes the Schwartz's space, i.e. $f \in S$ if and only if f is a C^∞ -function and f and all its derivatives are rapidly decreasing functions. We write s for the space of rapidly decreasing sequences. The Fourier transform of a function f is denoted by $\hat{f}$. We use the following standard notation for special functions: the Laguerre polynomials are denoted by $L_n^\alpha(t) = \sum_{k=0}^n \binom{n+\alpha}{n-k} (-t)^k / k!$, $\alpha > -1$, and the Bessel function of first kind by J_0 . We write D for the unit disc $D = \{w \in \mathbb{C} : |w| < 1\}$ and Ω_0 for the disc $\Omega_0 = \{u \in \mathbb{C} : |u - \frac{1}{2}| < \frac{1}{2}\}$.

1. PRELIMINARIES

We shall consider the space

$$S^+ = \{\phi : (0, +\infty) \rightarrow \mathbb{C} \text{ such that } \phi \text{ is a } C^\infty\text{-function and} \\ \|\phi\|_{k,n} = \sup_{t \in (0, +\infty)} t^k |\phi^{(n)}(t)| < +\infty \text{ for all } k, n \in \mathbb{N}\}$$

It is easy to verify that $(S^+, \|\cdot\|_{k,n}, k, n \in \mathbb{N})$ is a Fréchet space, and that $\phi \in S^+$ if and only if ϕ is the restriction to $(0, +\infty)$ of a function of the Schwartz's space S .

Also, we shall consider the following space: $S^+ \cap S = \{\phi \in S : \phi(t) = 0 \text{ for } t < 0\}$, which can be considered as a subspace of S^+ , in a natural way. It is

easy to prove that if $\phi \in S^+$ then $\phi \in S^+ \cap S$ if and only if $\lim_{t \rightarrow 0^+} \phi^{(k)}(t) = 0$ for all $k \geq 0$.

In [3, Th. 2.9] or [7, p. 548], are defined the Fourier-Laguerre coefficients of a function $\phi \in S^+$ by $a_n = \int_0^{+\infty} \phi(t) L_n(t) e^{-t/2} dt$ with $n \geq 0$ and there is proved the following theorem:

Theorem A. *The mapping $\mathcal{L}: S^+ \rightarrow s$ defined by*

$$\mathcal{L}(\phi) = (a_n)_n = \left(\int_0^{+\infty} \phi(t) L_n(t) e^{-t/2} dt \right)_n$$

is an isomorphism from S^+ onto s .

In [3, Th. 3.2], we characterize the Fourier-Laguerre coefficients of the functions of the space $S^+ \cap S$ in the following way:

Theorem B. *Let $\phi \in S^+$ and $a_n = \int_0^{+\infty} \phi(t) L_n(t) e^{-t/2} dt$. Then $\phi \in S^+ \cap S$ if and only if $\sum_{n=0}^{\infty} n^k a_n = 0$ for all $k \geq 0$.*

Also in [3, Th. 3.5] we prove the following theorem that we shall use later:

Theorem C. *The integral operator $\mathcal{H}_0: S^+ \rightarrow S^+$, defined by*

$$\mathcal{H}_0(\phi)(x) = \frac{1}{2} \int_0^{\infty} \phi(t) J_0(\sqrt{x}t) dt$$

is an isomorphism of S^+ onto itself, and $\mathcal{H}_0^2$ is the identity map. Moreover, if $(a_n)_n$ are the Fourier-Laguerre coefficients of ϕ then $((-1)^n a_n)_n$ are the Fourier-Laguerre coefficients of $\mathcal{H}_0(\phi)$.

The operator $\mathcal{H}_0$ is called a Hankel transform of degree 0.

2. SOME ANALYTIC FUNCTIONS SPACES RELATED TO S^+

Let $\phi \in L^1([0, +\infty))$, and consider the following analytic function $\tilde{\phi}$ in the lower half plane:

$$\tilde{\phi}(z) = \int_0^{+\infty} \phi(t) e^{-2\pi i z t} dt.$$

(Notice that if $z \in \mathbf{R}$ then $\tilde{\phi}(z) = \hat{\phi}(z)$.) Using the bilinear transformation $W(z) = (-\frac{1}{2} + 2\pi i z)/(\frac{1}{2} + 2\pi i z)$, which transforms the lower half plane in the unit disc, we obtain the following analytic function $\check{\phi}$ in the unit disc:

$$\check{\phi}(w) = \tilde{\phi}(Z(w)) = \int_0^{\infty} \phi(t) e^{-\frac{1}{2} \left(\frac{1+w}{1-w} \right) t} dt.$$

In the following theorem we relate the function $\check{\phi}$ with the Fourier-Laguerre coefficients of ϕ , for $\phi \in S^+$.

Theorem 2.1. Let $\phi \in S^+$ and $a_n = \int_0^\infty \phi(t) L_n(t) e^{-t/2} dt$. Then $\check{\phi}(w) = (1-w) \sum_{n=0}^\infty a_n w^n$ for $|w| < 1$.

Proof. As $(L_n(t) e^{-t/2})_n$ is an orthonormal system in $L^2([0, +\infty))$, $\phi(t) = \sum_n a_n L_n(t) e^{-t/2}$ and the series converges in $L^2([0, +\infty))$. Since $e^{-2\pi izt} \in L^2([0, +\infty))$ if $\Im z < 0$, we obtain $\tilde{\phi}(z) = \sum_n a_n \int_0^\infty L_n(t) e^{-t/2} e^{-2\pi izt} dt$. From [6, p. 191] we get $\tilde{\phi}(z) = \sum_n a_n (-\frac{1}{2} + 2\pi iz)^n / (\frac{1}{2} + 2\pi iz)^{n+1}$. If we put $w = (-\frac{1}{2} + 2\pi iz) / (\frac{1}{2} + 2\pi iz)$ we obtain $\check{\phi}(w) = (1-w) \sum_n a_n w^n$.

Considering Theorems A, B and 2.1 we obtain the following characterizations:

Theorem 2.2. Let $f \in H(D)$. The following conditions are equivalent:

- (a) There exists $\phi \in S^+$ such that $f(w) = \check{\phi}(w)$.
- (b) $f(w)/(1-w)$ and all its derivatives are bounded on D .

Theorem 2.3. Let $f \in H(D)$. The following conditions are equivalent:

- (a) There exists $\phi \in S^+ \cap S$ such that $f(w) = \check{\phi}(w)$.
- (b) $f(w)/(1-w)$ and all its derivatives are bounded on D and $\lim_{w \rightarrow 1} (f(w)/(1-w))^{(k)} = 0$ for all $k \geq 0$.

Let $\phi \in S^+$. Given $\phi(t) e^{t/2}$, in a similar way as we did for ϕ , we can define two functions $(\phi(t) e^{t/2})^\sim(z)$ and $(\phi(t) e^{t/2})^\sim(u)$, which are analytic in $\{z \in \mathbb{C}: \Im z < -1/4\pi\}$ and $\Omega_0 = \{u \in \mathbb{C}: |u - \frac{1}{2}| < \frac{1}{2}\}$ respectively.

Theorem 2.4. Let $f \in H(\Omega_0)$. The following conditions are equivalent:

- (a) There exists $\phi \in S^+ \cap S$ such that $(\phi(t) e^{t/2})^\sim(u) = f(u)$.
- (b) $f(u)/(1-u)$ and all its derivatives are bounded on Ω_0 and $\lim_{u \rightarrow 1} (f(u)/(1-u))^{(k)} = 0$ for all $k \geq 0$.

Proof. (a) $\Rightarrow$ (b). We define $\psi(t) = \phi(t) e^{t/2}$ and $g(z) = \tilde{\psi}(z - i/4\pi)$. It is clear that $g(z) = \tilde{\phi}(z)$. Let $h \in H(\Omega_0)$ defined by $h(u) = \check{\phi}(3 - (4/(u+1)))$ (it is well defined because $w(u) = 3 - (4/(u+1))$ transforms Ω_0 in D). By Theorem 2.3 (and because $d^k w(u)/du^k$ is bounded in Ω_0 for all $k \geq 0$) $h(u)/(1-u)$ and all its derivatives are bounded in Ω_0 and $\lim_{u \rightarrow 1} (h(u)/(1-u))^{(k)} = 0$ for all $k \geq 0$. But

$$h(u) = \check{\phi}(w(u)) = \tilde{\phi}(Z(w(u))) = g(Z(w(u))) = \tilde{\psi}\left(\frac{1}{4\pi i} \frac{1+u}{1-u}\right) = \tilde{\psi}(u) = f(u)$$

and the proof is finished.

(b) $\Rightarrow$ (a). Follows similarly because $u(w) = 4/(3-w) - 1$ transforms D in Ω_0 and $d^k u(w)/dw^k$ is bounded in D for all $k \geq 0$.

We need the next theorem, whose proof is suggested by Gardin's proof of a classical Borel's Theorem (see [4, p. 50]).

Theorem 2.5. Let $(a_n)_n$ be a sequence of complex numbers. There exists a function $f \in H(\Omega_0)$ such that:

- (a) f and all its derivatives are bounded in Ω_0 .
- (b) $\lim_{u \rightarrow 0} f^{(n)}(u) = a_n$ for all $n \geq 0$.
- (c) $\lim_{u \rightarrow 1} f^{(n)}(u) = 0$ for all $n \geq 0$.

We need two lemmas:

Lemma 1. There exists an analytic function $F: \Omega_0 \rightarrow \mathbb{C}$ satisfying:

- (a) F and all its derivatives are bounded in Ω_0 .
- (b) $\lim_{u \rightarrow 0} F^{(n)}(u) = 0$ for all $n > 0$ and $\lim_{u \rightarrow 0} F(u) = 1$.
- (c) $\lim_{u \rightarrow 1} F^{(n)}(u) = 0$ for all $n \geq 0$.

Proof. Let $\phi \in S^+ \cap S$ and $a_n = \int_0^\infty \phi(t) L_n(t) e^{-t/2} dt$. By Theorem 2.3, $\check{\phi}(w)$ and all its derivatives are bounded in D and $\lim_{w \rightarrow 1} \check{\phi}^{(n)}(w) = 0$ for all $n \geq 0$. We can choose ϕ such that $a_0 \neq 0$ and $\sup_{w \in D} |\check{\phi}(w)| = 1$, and there exists $z_0 \neq 1$, $|z_0| = 1$ such that $\lim_{w \rightarrow z_0} \check{\phi}(w) = 1$. Let $b_0 = 0$ and $b_n = a_n$ if $n \geq 1$. We consider the function $h(w) = \sum_n b_n w^n$. Since $(b_n) \in s$, h and all its derivatives are bounded in D . Also $\lim_{w \rightarrow 0} h(w) = 0$ and by the choice of the coefficients (notice that by Theorem B $\sum_n n^k a_n = 0$ for all $k \geq 0$) $\lim_{w \rightarrow 1} h(w) = -a_0 \neq 0$ and $\lim_{w \rightarrow 1} h^{(n)}(w) = 0$ for all $n \geq 1$.

Let $g: D \rightarrow \mathbb{C}$ be defined by $g(w) = h(\check{\phi}(w))$ (it is well defined because $|\check{\phi}(w)| \leq 1$ for all $w \in D$). Notice that g and all its derivatives are bounded in D .

We consider a bilinear application H , which transforms Ω_0 in D , sending 0 into z_0 and 1 into 1.

Finally, we take $F: \Omega_0 \rightarrow \mathbb{C}$ defined by $F(u) = -(1/a_0)g(H(u))$.

Lemma 2. Let $u \in \Omega_0$. Given n , let z be such that $z^{2^n} = u$ and $|\arg(z)|$ is the smallest possible. Then $z \in \Omega_0$.

Proof. It is enough to prove the Lemma for $|u - \frac{1}{2}| = \frac{1}{2}$. We then write $u = 1/(1 + \tan^2(\theta)) + (\tan \theta / (1 + \tan^2 \theta))i$ with $-\pi/2 < \theta < \pi/2$, so that $|z - \frac{1}{2}| \leq \frac{1}{2}$ is equivalent to $(\cos(\theta))^{1/2^n} \leq \cos(\theta/2^n)$, and we proceed by induction.

Proof (of Theorem 2.5). Let $(a_n)_n$ be a sequence of complex numbers. We define the sequence $(b_n)_n$ as $b_n = 1/2^{k_n}$ with $k_n = \min\{m: 1/2^m < 1/([|a_n|] + 1)\}$ (where $[x]$ is the integral part of the real number x). We define the function $f: \Omega_0 \rightarrow \mathbb{C}$ as

$$f(u) = \sum_{n=0}^{\infty} \frac{a_n}{n!} u^n F(u^{b_n}),$$

where F is the same function in Lemma 1 and we choose u^{b_n} as in Lemma 2 (with this choice u^{b_n} is an analytic function in Ω_0 , and take its values in Ω_0).

We prove that f and all its derivatives are bounded in Ω_0 .

Since $\lim_{u \rightarrow 1} F^{(n)}(u) = 0$ for all $n \geq 0$, there exists $M_1 > 0$ such that $|F(u)| \leq M_1 |1 - u|^2$. Let u be such that $|u - \frac{1}{2}| = \frac{1}{2}$; we can write $u = 1/(1 + \tan^2(\theta)) + (\tan \theta / (1 + \tan^2(\theta)))i$ with $-\pi/2 < \theta < \pi/2$. It is clear that $|1 - u|^2 = \sin^2(\theta)$ and $1 - |u| = 1 - \cos(\theta)$. So $|F(u)| \leq M(1 - |u|)$ for $u \in \Omega_0$.

The following shows that f is analytic and bounded in Ω_0 :

$$(1) \quad \left| \frac{a_n}{n!} u^n F(u^{b_n}) \right| \leq \frac{|a_n|}{n!} |u|^n M(1 - |u|^{b_n}) \leq M \frac{|a_n|}{n!} b_n |u|^{n+b_n-1} \leq M \frac{1}{n!}$$

for all $u \in \Omega_0$ and $n \geq 1$;

and from $n = 0$ $|F(u_{b_0})| \leq \text{constant}$.

We shall study the derivatives of f .

We have:

$$(2) \quad \begin{aligned} f^{(k)}(u) &= \sum_{n=k}^{\infty} \frac{a_n}{n!} n(n-1) \cdots (n-k+1) u^{n-k} F(u^{b_n}) \\ &+ \sum_{m=1}^k \binom{k}{m} \sum_{n=k-m}^{\infty} \frac{a_n}{n!} n(n-1) \cdots (n-k+m+1) u^{n-k+m} \frac{d^m F(u^{b_n})}{du^m} \\ &= \sum_{n=0}^{\infty} \frac{a_{n+k}}{n!} u^n F(u^{b_{n+k}}) + \sum_{m=1}^k \binom{k}{m} \sum_{n=0}^{\infty} \frac{a_{n+k-m}}{n!} u^n \frac{d^m F(u^{b_{n+k-m}})}{du^m}. \end{aligned}$$

Proceeding as in (1), we obtain

$$(3) \quad \left| \frac{a_{n+k}}{n!} u^n F(u^{b_{n+k}}) \right| \leq M_2 \frac{1}{n!} \quad \text{if } u \in \Omega_0 \text{ for certain } M_2 > 0$$

We can write (see [11, p. 12 (83)])

$$\begin{aligned} u^n \cdot \frac{d^m F(u^{b_{n+k-m}})}{du^m} &= u^n \cdot \sum_{l=1}^m \frac{(-1)^l}{l!} \sum_{j=1}^l (-1)^j \binom{l}{j} u^{(l-j)b_{n+k-m}} j b_{n+k-m} \\ &\quad (j b_{n+k-m} - 1) \cdots (j b_{n+k-m} - m + 1) u^{j b_{n+k-m} - m} F^{(l)}(u^{b_{n+k-m}}) \\ &= b_{n+k-m} \sum_{l=1}^m \frac{(-1)^l}{l!} \sum_{j=1}^l (-1)^j \binom{l}{j} j (j b_{n+k-m} - 1) \cdots \\ &\quad \cdots (j b_{n+k-m} - m + 1) u^{l b_{n+k-m}} \frac{F^{(l)}(u^{b_{n+k-m}})}{u^{m-n}}. \end{aligned}$$

If $n \geq m$, since F and all its derivatives are bounded in Ω_0 , we obtain $|F^{(l)}(u^{b_{n+k-m}})/u^{m-n}| \leq C_1$ if $u \in \Omega_0$ (the constant only depends on l).

If $0 \leq n < m$, let $\beta = \max\{[(m-n)/b_{n+k-m}] + 1 : 0 \leq n < m\}$. Since $\lim_{u \rightarrow 0} F^{(\alpha)}(u) = 0$ if $\alpha \geq 1$, there exists $C_2 > 0$, (which only depends on l, m and k) such that $|F^{(l)}(u)| \leq C_2 |u|^\beta$ if $u \in \Omega_0$. So

$$\left| \frac{F^{(l)}(u^{b_{n+k-m}})}{u^{m-n}} \right| \leq C_2 \frac{|u|^{\beta b_{n+k-m}}}{|u|^{m-n}} \leq C_2$$

if $u \in \Omega_0$. So, for $n \geq 0$, $|F^{(l)}(u^{b_{n+k-m}})/u^{m-n}| \leq \text{constant}$ if $u \in \Omega_0$, and the constant only depends on l, m and k . Hence $|u^n \cdot d^m F(u^{b_{n+k-m}})/du^m| \leq M_3 b_{n+k-m}$ if $u \in \Omega_0$, where M_3 is a positive constant, $m = 1, \dots, k$ and $n \geq 0$.

We obtain

$$(4) \quad \left| \frac{a_{n+k-m}}{n!} u^n \frac{d^m F(u^{b_{n+k-m}})}{du^m} \right| \leq \frac{M_3}{n!} \quad \text{if } u \in \Omega_0.$$

Using (2), (3) and (4), we get $|f^{(k)}(u)| \leq \text{constant}$ if $u \in \Omega_0$.

By the properties of F and (2), $\lim_{u \rightarrow 0} f^{(k)}(u) = a_k$ and $\lim_{u \rightarrow 1} f^{(k)}(u) = 0$ for all $k \geq 0$, and the proof is finished.

In [14] using other techniques, we can find theorems of the same kind as Theorem 2.5 for more general sets Ω , but the function f which interpolates the sequence $(a_n)_n$ does not satisfy condition (c).

3. THE STIELTJES MOMENTS PROBLEM IN THE SPACE $S^+ \cap S$

We prove that the Stieltjes moments problem always has a solution in the space $S^+ \cap S$ (and so in the space S^+).

Theorem 3.1. *Let $(a_n)_n$ be a sequence of complex numbers. Then there exists a function $\phi \in S^+ \cap S$ such that $\int_0^\infty t^n \phi(t) dt = a_n$ for all $n \geq 0$.*

Proof. We define the following sequence $b_n = \sum_{k=0}^n ((-1)^k/k!) \binom{n}{k} a_k$. As $L_n(t) = \sum_{k=0}^n ((-1)^k/k!) \binom{n}{k} t^k$, if there exists $\phi \in S^+ \cap S$ such that $\int_0^\infty \phi(t) L_n(t) dt = b_n$, it is clear that $\int_0^\infty \phi(t) t^n dt = a_n$. Let $c_n = n! b_n$. By Theorems 2.4 and 2.5, there exists $f \in H(\Omega_0)$ and $\phi \in S^+ \cap S$ such that $g(u) = (1-u)f(u) = (\phi e^{1/2})(u)$ and $\lim_{u \rightarrow 0} f^{(n)}(u) = c_n$. We shall prove that $b_n = \int_0^{+\infty} \phi(t) L_n(t) dt$. Indeed

$$\begin{aligned} g(u) &= \int_0^{+\infty} \phi(t) e^{-t\left(\frac{u}{1-u}\right)} dt \\ &= (1-u) \int_0^{+\infty} \phi(t) \frac{1}{1-u} e^{-t\left(\frac{u}{1-u}\right)} dt \\ &= (1-u) \int_0^{+\infty} \varphi(t, u) dt. \end{aligned}$$

So $f(u) = \int_0^{+\infty} \varphi(t, u) dt$.

By [6, p. 189] $\varphi(t, u) = \phi(t) \sum_{k=0}^\infty L_k(t) u^k$, for $t \in \mathbf{R}$ and $|u| < 1$.

Formally

$$f^{(n)}(u) = \int_0^{+\infty} \phi(t) \sum_{k=n}^\infty k(k-1) \cdots (k-n+1) L_k(t) u^{k-n} dt$$

and

$$\lim_{u \rightarrow 0} f^{(n)}(u) = \int_0^{+\infty} \phi(t) n! L_n(t) dt,$$

and as $\lim_{u \rightarrow 0} f^{(n)}(u) = c_n$ we deduce that $b_n = \int_0^{+\infty} \phi(t) L_n(t) dt$.

Now it suffices to see that $|\varphi(t, u)| \leq h(t)$ with $h \in L^1([0, +\infty))$. As

$$\begin{aligned}\varphi(t, u) &= \phi(t) \frac{1}{1-u} \exp\left(-t \frac{u}{1-u}\right) \\ &= \phi(t) \frac{1}{1-u} \exp\left(-t \frac{(x+iy)(1-x+iy)}{(1-x)^2+y^2}\right)\end{aligned}$$

with $u = x + iy$, we have $|\varphi(t, u)| \leq |\phi(t)| \frac{1}{1-u} \exp\left(-t \frac{x(1-x)-y^2}{(1-x)^2+y^2}\right)$.

So is enough to take $u \in \Omega_0$ with $\Re u < \frac{1}{2}$.

We deduce the existence of solution for a generalized Stieltjes moments problem.

Corollary 3.2. *Let $(a_n)_n$ be an sequence of complex numbers and let $(k_n)_n$ be an arithmetic sequence of real numbers (i.e. $k_n = a + bn$ for $a, b \in \mathbf{R}$). Then there exists a function $\phi \in S^+ \cap S$ such that $\int_0^\infty t^{k_n} \phi(t) dt = a_n$ for all $n \geq 0$.*

Proof. By the previous theorem, there exists $\phi \in S^+ \cap S$ such that $\int_0^\infty t^n \phi(t) dt = a_n$ for all $n \geq 0$. Setting $t = x^b$, we get $a_n = \int_0^\infty |b|x^{-a+b-1} \phi(x^b) x^{k_n} dx$. Notice that as $\phi \in S^+ \cap S$ then $\lim_{t \rightarrow 0} \phi^{(k)}(t) = 0$ for all $k \geq 0$, so $\psi(x) = |b|x^{-a+b-1} \phi(x^b) \in S^+ \cap S$ and $\psi(x)x^{k_n} \in L^1([0, +\infty))$, which proves the theorem.

4. FUNCTIONS IN S^+ WITH NULL MOMENTS

In the following theorem, we characterize the functions of S^+ such that their moments are null.

Theorem 4.1. *Let $\phi \in S^+$, and $a_n = \int_0^\infty \phi(t) L_n(t) e^{-t/2} dt$. The following conditions are equivalent:*

- (a) $\int_0^\infty t^k \phi(t) dt = 0$ for all $k \geq 0$.
- (b) $\sum_n (-1)^n n^k a_n = 0$ for all $k \geq 0$.
- (c) *There exists $\psi \in S^+ \cap S$ such that ϕ is the Hankel transform of degree 0 of ψ , i.e.*

$$\phi(t) = \mathcal{H}_0(\psi)(t) = \frac{1}{2} \int_0^\infty \psi(x) J_0(\sqrt{xt}) dx.$$

Proof. (a) $\Rightarrow$ (b). It is clear that $\int_0^\infty \phi(t) t^k dt = 0$ for all $k \geq 0$ if and only if $\lim_{z \rightarrow 0} \tilde{\phi}^{(k)}(z) = 0$ for all $k \geq 0$ and so if and only if $\lim_{w \rightarrow -1} \check{\phi}^{(k)}(w) = 0$ for all $k \geq 0$ (because $\check{\phi}(w) = \tilde{\phi}(Z(w))$ and $d^n Z(w)/dw^n$ is bounded near $w = -1$). It is clear that $\lim_{w \rightarrow -1} \check{\phi}^{(k)}(w) = 0$ for all $k \geq 0$ if and only if $\lim_{w \rightarrow -1} (\check{\phi}(w)/(1-w))^{(k)} = 0$. We have $\check{\phi}(w) = (1-w) \sum_n a_n w^n$ and

$(a_n)_n \in s$, so $\check{\phi}(w)/(1-w) = \sum_n a_n w^n$ and

$$\left(\frac{\check{\phi}(w)}{(1-w)} \right)^{(k)} = \sum_n n^k a_n w^{n-k} + b_{k-1}^{(k)} \sum_n n^{k-1} a_n w^{n-k} + \cdots + b_1^{(k)} \sum_n n a_n w^{n-k}$$

for certain $b_{k-1}^{(k)}, \dots, b_1^{(k)} \in \mathbf{N}$ and $k \geq 1$. Applying successively Theorems of Tauber and Abel, we get $\int_0^\infty \phi(t) t^k dt = 0$ if and only if $\sum_n (-1)^n n^k a_n = 0$ for all $k \geq 0$.

(b) $\Leftrightarrow$ (c). By Theorem C the Fourier-Laguerre coefficients of $\mathcal{H}_0(\phi)$ are $((-1)^n a_n)_n$. So by Theorem B $\sum_n (-1)^n n^k a_n = 0$ if and only if $\mathcal{H}_0(\phi) \in S^+ \cap S$. Since $\mathcal{H}_0^2 = \text{id}$, setting $\psi = \mathcal{H}_0(\phi)$ we finish the proof.

We shall extend the previous theorem for the generalized moment problem.

In [3] Theorem A is generalized in the following way:

Let $\alpha > -1$, $t^{\alpha/2} S^+ = \{t^{\alpha/2} \phi: \phi \in S^+\}$ and $\mathcal{L}_n^\alpha(t) = \tau_n L_n^\alpha(t) t^{\alpha/2} e^{-t/2}$, where $\tau_n = (n!/\Gamma(n+\alpha+1))^{1/2}$ and $L_n^\alpha(t)$ are the generalized Laguerre polynomials. It is well known that $(\mathcal{L}_n^\alpha(t))_n$ is an orthonormal system in $L^2([0, +\infty))$. If $\psi \in t^{\alpha/2} S^+$ we define the Fourier-Laguerre coefficients of ψ as $a_n = \int_0^\infty \psi(t) \mathcal{L}_n^\alpha(t) dt$. We prove in [3] the following theorem:

Theorem D. *The mapping $\mathcal{L}: t^{\alpha/2} S^+ \rightarrow s$ defined by*

$$\mathcal{L}(\phi) = (a_n)_n = \left(\int_0^{+\infty} \phi(t) \mathcal{L}_n^\alpha(t) dt \right)_n$$

is an isomorphism of the space $t^{\alpha/2} S^+$ onto s .

The proofs of the following theorems are analogous to the proofs of Theorems 2.1 and 2.2.

Theorem 4.2. *Let $\phi \in S^+$ and $a_n = \int_0^\infty t^{\alpha/2} \phi(t) \mathcal{L}_n^\alpha(t) dt$. Then $(t^\alpha \phi)^\sim(w) = (1-w)^{\alpha+1} \sum_{n=0}^\infty a_n \tau_n^{-1} w^n$.*

Theorem 4.3. *Let $f \in H(D)$. The following are equivalent:*

- (a) *There exists $\phi \in S^+$ such that $f(w) = (t^\alpha \phi)^\sim(w)$.*
- (b) *$f(w)/(1-w)^{\alpha+1}$ and all its derivatives are bounded on D .*

Applying these results, we obtain:

Theorem 4.4. *Let $\alpha > 0$ and $\phi \in S^+$. Then $\int_0^\infty t^{\alpha+n} \phi(t) dt = 0$ if and only if there exists $\psi \in S^+ \cap S$ such that $\phi(t) = (1/t^\alpha)(\mathcal{H}_0(\psi) * (x^{\alpha-1} e^{-x/2}/\Gamma(\alpha)))(t)$.*

Proof. Proceeding as in the proof of Theorem 4.1 we obtain $\int_0^\infty t^{\alpha+n} \phi(t) dt = 0$ if and only if $\lim_{w \rightarrow -1} (t^\alpha \phi)^{(k)}(w) = 0$ for all $k \geq 0$. So $\lim_{w \rightarrow -1} ((t^\alpha \phi)^\sim(w)/(1-w)^{\alpha+1})^{(k)}(w) = 0$ for all $k \geq 0$.

By Theorem 4.2 $(t^\alpha \phi)(w)/(1-w)^{\alpha+1} = \sum_n b_n w^n$ with $(b_n) \in s$, so $\int_0^\infty t^{\alpha+n} \phi(t) dt = 0$ if and only if $\sum_n (-1)^n n^k b_n = 0$ for all $k \geq 0$ (see proof of Theorem 4.1).

By Theorems A and 4.1, we obtain $\int_0^\infty t^{\alpha+n} \phi(t) dt = 0$ if and only if $(t^\alpha \phi)(w) = (1-w)^\alpha (\mathcal{H}_0(\psi))(w)$ where $\psi \in S^+ \cap S$. If we put $w = (-\frac{1}{2} + 2\pi iz)/(\frac{1}{2} + 2\pi iz)$, it follows that

$$\widetilde{t^\alpha \phi}(z) = \frac{1}{(\frac{1}{2} + 2\pi iz)^\alpha} \widetilde{\mathcal{H}_0(\psi)}(z).$$

If $z \in \mathbf{R}$ and noticing that $(x^{\alpha-1} e^{-x/2}/\Gamma(\alpha))^\wedge(z) = 1/(\frac{1}{2} + 2\pi iz)^\alpha$ (see [5, p. 144]) we get

$$\widehat{t^\alpha \phi}(z) = (x^{\alpha-1} e^{-x/2}/\Gamma(\alpha))^\wedge(z) \widehat{\mathcal{H}_0(\psi)}(z).$$

So $t^\alpha \phi(t) = ((x^{\alpha-1} e^{-x/2}/\Gamma(\alpha)) * \mathcal{H}_0(\psi))(t)$ and the theorem is proved.

We study the functions of $S^+ \cap S$ with null moments.

Let $T = \{\phi \in S^+ \cap S : \int_0^\infty t^k \phi(t) dt = 0 \text{ for all } k \geq 0\}$. From Theorems B, C and 4.1 we get:

Theorem 4.5. *Let $\phi \in S^+$ and $a_n = \int_0^\infty \phi(t) L_n(t) e^{-t/2} dt$. Then $\phi \in T$ if and only if $\sum_n n^k a_n = 0$ and $\sum_n (-1)^n n^k a_n = 0$ for all $k \geq 0$. So $T = (S^+ \cap S) \cap \mathcal{H}_0(S^+ \cap S)$ and $\mathcal{H}_0$ is an isomorphism of T onto itself.*

We give examples of non-null functions in the spaces T .

Indeed, let $\phi_1, \phi_2 \in S^+ \cap S$ and consider $\phi_3 = \phi_1 * \mathcal{H}_0(\phi_2)$. It is clear that $\check{\phi}_3(w) = \check{\phi}_1(w)(\mathcal{H}_0(\phi_2))(w)$. Let $a_n^i = \int_0^\infty \phi_i(t) L_n(t) e^{-t/2} dt$. By Theorem 2.1 $\check{\phi}_i(w) = (1-w) \sum_n a_n^i w^n$ for $i = 1, 2, 3$.

Since $\phi_1 \in S^+ \cap S$, by Theorem B $\lim_{w \rightarrow -1} \check{\phi}_1^{(k)}(w) = 0$ for all $k \geq 0$ and since $\phi_2 \in S^+ \cap S$ by Theorem 4.1, $\lim_{w \rightarrow -1} (\mathcal{H}_0 \phi_2)^{(k)}(w) = 0$ for all $k \geq 0$. So $\lim_{w \rightarrow -1} \check{\phi}_3^{(k)}(w) = \lim_{w \rightarrow -1} \check{\phi}_1^{(k)}(w) = 0$ for all $k \geq 0$, hence (see proof of Theorem 4.1) $\sum_n n^k a_n^3 = 0$ and $\sum_n (-1)^n n^k a_n^3 = 0$ for all $k \geq 0$. By the previous theorem, $\phi_3 \in T$.

Finally, we propose the following problem (converse of the previous example).

Problem. Let $\phi \in T$, do there exist $\phi_1, \phi_2 \in S^+ \cap S$ such that $\phi = \phi_1 * \mathcal{H}_0(\phi_2)$?

Notice that this problem is equivalent to the following one:

Let $f \in H(D)$ such that f and all its derivatives are bounded in D and $\lim_{w \rightarrow -1} f^{(k)}(w) = \lim_{w \rightarrow -1} f^{(k)}(w) = 0$ for all $k \geq 0$. Do there exist two functions $g, h \in H(D)$ bounded and with all its derivatives bounded in D such that $\lim_{w \rightarrow -1} g^{(k)}(w) = 0$ and $\lim_{w \rightarrow -1} h^{(k)}(w) = 0$ for all $k \geq 0$ and $f = g \cdot h$?

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