EXTENSION OF OPERATORS FROM SUBSPACES OF $c_0(\Gamma)$ INTO $C(K)$ SPACES

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ABSTRACT. It is shown that for every $\varepsilon > 0$, every bounded linear operator T from a subspace X of $c_0(\Gamma)$ into a $C(K)$ space has an extension T from $c_0(\Gamma)$ into the $C(K)$ space such that $\|T\| \leq (1 + \varepsilon)\|T\|$. Even when Γ is countable, T is compact, and X has codimension 1 in c_0 , the “ ε ” cannot be replaced by 0. These results answer questions raised by J. Lindenstrauss and A. Pełczyński in 1971.

J. Lindenstrauss and A. Pełczyński proved [LP, Theorem 3.1] that for every $\varepsilon > 0$, every operator T from a subspace of c_0 into a $C(K)$ space has an extension T from c_0 into the $C(K)$ space such that $\|T\| \leq (1 + \varepsilon)\|T\|$. They asked in [LP] whether this result remains true if c_0 is replaced with $c_0(\Gamma)$ with Γ an uncountable index set. In Theorem 4 we give an affirmative answer to this problem.

Another question in [LP] was whether in their Theorem 3.1 “ ε ” can be taken to be 0; i.e., whether the extension T of T can be chosen to have the same norm as T . In addition to the inherent interest in such an improvement of their theorem, the authors suggested that such a result should lead to a positive solution to their nonseparable problem solved here. The answer to their question is, however, negative: Example 6 is a compact operator from a codimension 1 subspace of c_0 into a $C(K)$ space which has no norm-preserving extension to an operator from c_0 into the $C(K)$ space.

We use standard Banach space theory terminology as may be found, e.g., in [LT].

The proof of Theorem 4 is based on the following simple proposition from [Z], which is in turn based on one of Lindenstrauss’s approaches [L] to extension problems:

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Proposition 1. *Let $\lambda \geq 1$ and let X be a subspace of a Banach space Y . Then the following are equivalent:*

- (1) *For every compact Hausdorff space K and every operator $T: X \rightarrow C(K)$ there is an extension $\mathbf{T}: Y \rightarrow C(K)$ with $\|\mathbf{T}\| \leq \lambda \|T\|$.*
- (2) *There is a weak*-continuous function $\psi: \text{Ball}(X^*) \rightarrow \lambda \text{Ball}(Y^*)$ such that $\psi(x^*)(x) = x^*(x)$ for every $x \in X$ and $x^* \in \text{Ball}(X^*)$.*

We also need an easy decomposition lemma for subspaces of $c_0(\Gamma)$ which may be known but for which we have no reference.

Lemma 2. *Let Γ be an uncountable set and let X be a subspace of $c_0(\Gamma)$. Then Γ can be decomposed into a family $\{\Gamma_\alpha\}_{\alpha \in A}$ of pairwise disjoint countable sets such that if $\alpha \in A$ and $X_\alpha = \{x \in X: \text{support}(x) \subset \Gamma_\alpha\}$, then for every $x \in X$, the restriction $x|_{\Gamma_\alpha}$ of x to Γ_α is in X_α .*

Proof. For a subset Δ of Γ , let R_Δ denote the restriction operator from $c_0(\Gamma)$ onto $c_0(\Delta)$, defined by $R_\Delta(x) = x|_\Delta$. We must find a decomposition $\{\Gamma_\alpha\}_{\alpha \in A}$ of Γ into pairwise disjoint countable sets so that $R_{\Gamma_\alpha}[X] \subset X$ for every $\alpha \in A$. Now if $\{\Delta_\beta\}_{\beta \in B}$ are pairwise disjoint subsets of Γ , then the series $\sum_{\beta \in B} R_{\Delta_\beta}$ converges strongly to the projection R_Δ , where $\Delta = \cup_{\beta \in B} \Delta_\beta$, so a standard exhaustion argument reduces the lemma to the following claim:

Claim. For each countable subset Δ_0 of Γ , there is a countable subset Ω of Γ such that $\Delta_0 \subset \Omega$ and $R_\Omega[X] \subset X$.

To prove the claim, construct countable sets $\Delta_0 \subset \Delta_1 \subset \Delta_2 \subset \dots$ and separable subspaces $X_0 \subset X_1 \subset X_2 \subset \dots$ of X so that for each $n = 0, 1, 2, \dots$,

- (a) $R_{\Delta_n}[\text{Ball}(X_n)]$ is dense in $R_{\Delta_n}[\text{Ball}(X)]$,
- (b) $X_n \subset c_0(\Delta_{n+1})$,

and set $\Omega = \cup_{n=1}^\infty \Delta_n$. Now let $x \in \text{Ball}(X)$ and select $x_n \in \text{Ball}(X_{n+1})$ so that $\|R_{\Delta_n}(x - x_n)\| \rightarrow 0$ as $n \rightarrow \infty$. Then also $\|R_\Omega x - R_{\Delta_n} x_n\| \rightarrow 0$ as $n \rightarrow \infty$. But the sequence $\{x_n - R_{\Delta_n} x_n\}_{n=1}^\infty$ is bounded and the terms are disjointly supported, so the sequence tends to 0 weakly in $c_0(\Gamma)$. Hence the sequence $\{x_n\}_{n=1}^\infty$ tends weakly to $R_\Omega x$, whence $R_\Omega x$ is in X . $\square$

Remark 3. The proof of Lemma 2 works in any space which has an extended shrinking unconditional basis. A more complicated argument gives a similar result for any space which has an extended shrinking Markushevich basis.

Theorem 4. *Let Γ be an uncountable set, let $\varepsilon > 0$, let X be a subspace of $c_0(\Gamma)$, and suppose that K is a compact Hausdorff space. Then every operator $T: X \rightarrow C(K)$ has an extension $\mathbf{T}: c_0(\Gamma) \rightarrow C(K)$ with $\|\mathbf{T}\| \leq (1 + \varepsilon)\|T\|$.*

Proof. Let $\{\Gamma_\alpha\}_{\alpha \in A}$ and $\{X_\alpha\}_{\alpha \in A}$ be as in Lemma 2, and identify X^* with the $\mathcal{L}_1(A)$ -sum of $\{X_\alpha^*\}_{\alpha \in A}$ in the natural way. Since each of the sets Γ_α is countable and $X_\alpha \subset c_0(\Gamma_\alpha)$, we get from Theorem 3.1 of [LP] and Proposition 1 weak*-continuous functions $\psi_\alpha: \text{Ball}(X_\alpha^*) \rightarrow (1 + \varepsilon)\text{Ball}(\mathcal{L}_1(\Gamma_\alpha))$ which extend functionals. Now we define $\psi: \text{Ball}(X^*) \rightarrow (1 + \varepsilon)\text{Ball}(\mathcal{L}_1(\Gamma))$ by setting

for $x^* \in \text{Ball}(X^*)$ $\psi(x^*) = \sum_{\alpha \in A} \psi_\alpha(x^*|_{X_\alpha})$. It is easily checked that ψ is weak*-continuous and extends functionals, so Proposition 1 gives the desired conclusion. $\square$

For Example 6, we use the following well-known analogue of Proposition 1.

Proposition 5. *Let $\lambda \geq 1$, let X be a subspace of a Banach space Y , and let K be a compact Hausdorff space. Then the following are equivalent:*

- (1) *Every compact operator $T: X \rightarrow C(K)$ has an extension $\mathbf{T}: Y \rightarrow C(K)$ with $\|\mathbf{T}\| \leq \lambda \|T\|$.*
- (2) *For every continuous function $\psi: K \rightarrow [\text{Ball}(X^*), \|\cdot\|_{X^*}]$ there is a continuous function $\psi: K \rightarrow [\lambda \text{Ball}(Y^*), \text{weak}^*]$ such that $\psi(k)(x) = \psi(k)(x)$ for every $x \in X$ and $k \in K$.*

Example 6. There is a compact operator from a hyperplane X of c_0 into $C(\mathbb{N} \cup \{\infty\})$ (the space of continuous functions from a convergent sequence) which has no norm-preserving extension to an operator from c_0 into $C(\mathbb{N} \cup \{\infty\})$.

Proof. Let $\{e_n\}_{n=1}^\infty$ (respectively, $\{F_n\}_{n=1}^\infty$) denote the unit vector basis of c_0 (respectively, $\mathcal{C}_1 = c_0^*$). Fix $0 < \lambda < \frac{1}{2}$ and set $F = \sum_{n=1}^\infty \lambda^{n-1} (F_{2n-1} + F_{2n})$. X is defined to be the null space of the linear functional F . Set

$$G = \sum_{n=1}^\infty \lambda^{n-1} F_{2n} - \sum_{n=2}^\infty \lambda^{n-1} F_{2n-1}$$

and, for $n = 2, 3, \dots$, let

$$G(n) = \sum_{i=1}^n \lambda^{i-1} F_{2i} - \sum_{i=2}^n \lambda^{i-1} F_{2i-1},$$

$$H(n) = \sum_{i=1}^n \lambda^{i-1} F_{2i} - \sum_{i=2}^n \lambda^{i-1} F_{2i-1} + \lambda^{n-1} (F_{2n+1} + F_{2n+2}).$$

Since in $\mathcal{C}_1$ $G(n) \rightarrow G$ and $H(n) \rightarrow G$ as $n \rightarrow \infty$, also in X^* $G(n)|_X \rightarrow G|_X$ and $H(n)|_X \rightarrow G|_X$ as $n \rightarrow \infty$. Set

$$\tilde{G}(n) = \|G(n)|_X\|^{-1} G(n)|_X,$$

$$\tilde{H}(n) = \|H(n)|_X\|^{-1} H(n)|_X,$$

$$\tilde{G} = \|G|_X\|^{-1} G|_X,$$

and let $K = \{\tilde{G}|_X\} \cup \{\tilde{G}(n)|_X\}_{n=1}^\infty \cup \{\tilde{H}(n)|_X\}_{n=1}^\infty$ be considered as a subset of X^* in the norm topology, so that K is homeomorphic to $\mathbb{N} \cup \{\infty\}$ and the inclusion mapping $\psi: K \rightarrow [\text{Ball}(X^*), \|\cdot\|_{X^*}]$ is continuous.

To complete the proof, it is sufficient by Proposition 5 to check that for each n , $G(n)|_X$ and $H(n)|_X$ have unique Hahn-Banach extensions, say, G_n and H_n , respectively, to linear functionals on c_0 , and that the sequences $\{G_n\}_{n=1}^\infty$

and $\{H_n\}_{n=1}^\infty$ have different weak* limits in $\mathcal{L}_1$. Now the minimum over all $t \in \mathbf{R}$ of the quantity

$$\|G(n) - tF\|_{\mathcal{L}_1} = |t| \left[1 + \frac{2\lambda^n}{1-\lambda} \right] + |1-t| \frac{1-\lambda^n}{1-\lambda} + \lambda |1+t| \frac{1-\lambda^{n-1}}{1-\lambda}$$

occurs only at $t = 0$, so $G_n \equiv G(n)$ is the unique Hahn-Banach extension of $G(n)|_X$. The minimum over all $t \in \mathbf{R}$ of the quantity

$$\|H(n) - tF\|_{\mathcal{L}_1} = |t| \left[1 + \frac{2\lambda^{n+1}}{1-\lambda} \right] + |1-t| \frac{1-\lambda^n}{1-\lambda} + \lambda |1+t| \frac{1-\lambda^{n-1}}{1-\lambda} + 2\lambda^{n-1} |1-\lambda t|$$

occurs only at $t = 1$, so $H_n \equiv H(n) - F$ is the unique Hahn-Banach extension of $H(n)|_X$. But, as mentioned at the beginning of the proof, the sequence $\{G_n\}_{n=1}^\infty$ converges in $\mathcal{L}_1$ to G and $\{H_n\}_{n=1}^\infty$ converges in $\mathcal{L}_1$ to $G - F$. $\square$

REFERENCES

- [L] J. Lindenstrauss, *Extension of compact operators*, Mem. Amer. Math. Soc., no. 48, Amer. Math. Soc., Providence, R.I., 1964.
- [LP] J. Lindenstrauss and A. Pełczyński, *Contributions to the theory of the classical Banach spaces*, J. Funct. Anal. **8** (1971), 225–249.
- [LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces I: sequence spaces*, Springer-Verlag, Berlin, 1977.
- [Z] M. Zippin, *The embedding of Banach spaces into spaces with structure*, Illinois J. Math. (to appear).

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