PRODUCTS OF PERFECTLY MEAGER SETS AND LUSIN'S FUNCTION

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ABSTRACT. In [R] Reclaw solved an old problem of Marczewski by constructing two perfectly meager sets with a non-perfectly meager cartesian product. We show another method for getting such sets.

§1

A subset S of a polish (i.e., complete separable metric) space X is *perfectly meager* if every dense in itself subset of S is meager in itself or, equivalently, if S is meager in each perfect subset of X ([K, p. 516; Mi, p. 213]). An uncountable subset of a polish space is a *Lusin set* if it has countable intersections with each meager set ([K, p. 525; Mi, p. 204]).

In 1935 Marczewski [Ma] (see also [Mi, p. 222]) asked whether a cartesian product of perfectly meager sets is perfectly meager. Recently Reclaw [R] showed that if a Lusin set exists, then the answer is negative. He constructed a closed subset D of $2^\omega \times 2^\omega$ such that both projections send Lusin subsets of D to perfectly meager sets. It follows that the projections of a Lusin subset of D provide an answer to Marczewski's question: they are perfectly meager and their product, containing a Lusin subset of D , is non-meager in D .

Sending Lusin sets to perfectly meager sets recalls a function constructed in 1933 by Lusin [L]. This is a continuous one-to-one function Φ from the space $\mathcal{N}$ of irrationals into itself which maps Lusin sets of $\mathcal{N}$ to perfectly meager sets. The function Φ proved to be very useful in generating counterexamples in the theory of Baire subsets of the reals. Sierpiński gives a number of them in his book [S, pp. 63–75]. In the present paper we add one more counterexample. Namely we show that a permutation of coordinates in the cartesian square $\Phi \times \Phi$ yields a negative answer to Marczewski's question.

Theorem. *The set $D = \{((x_0, \Phi(x_1)), (x_1, \Phi(x_0))) : x_0, x_1 \in \mathcal{N}\}$ is a closed, homeomorphic to $\mathcal{N}$, subset of $(\mathcal{N} \times \mathcal{N}) \times (\mathcal{N} \times \mathcal{N})$ such that both projections onto $\mathcal{N} \times \mathcal{N}$ send Lusin subsets of D to perfectly meager sets.*

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Note that our set D is the graph of a Borel automorphism of $\mathcal{N} \times \Phi[\mathcal{N}]$, in particular for each $(x, y) \in \mathcal{N} \times \mathcal{N}$ the sections $D \cap (\{(x, y)\} \times \mathcal{N}^2)$ and $(\mathcal{N}^2 \times \{(x, y)\}) \cap D$ consist of at most one point. It follows that the cartesian product of the projections of a Lusin subset of D does not have the Baire property in the restricted sense (see [S, p. 71, Proposition C20] or [K, p. 530, Theorem 3] for a similar reasoning).

The theorem will be proved in a slightly more abstract form in §2. In §3 we prove a version of the theorem independent of the existence of Lusin sets. We show (Proposition 3.3) a Borel automorphism of a polish space X such that its graph D is a closed subset of $X \times X$ with the property that each $E \subseteq X$ contains a subset G comeager in E with $(G \times X) \cup (X \times G)$ nowhere dense in D . (Reclaw's set D also has this property, but it is not the graph of a function.)

We close the paper by showing that Lusin's function can be viewed as the identity on $\mathbf{Q}^\omega$, the countable cartesian product of rationals, so our D is in fact the diagonal in $(\mathbf{Q}^\omega)^2 \times (\mathbf{Q}^\omega)^2$.

§2

We start with an easy but useful general observation. We say that a family $\mathcal{F}$ of continuous functions from polish spaces to polish spaces has *the lifting property* if for any function f from $\mathcal{F}$ and for any polish space Z we have

- (i) for any homeomorphism g from Z onto $\text{dom}(f)$ the composition $f \circ g$ is in $\mathcal{F}$,
- (ii) the lifting $\bar{f}$ of f to Z defined by $\bar{f}(z, x) = (z, f(x))$, $z \in Z, x \in \text{dom}(f)$, is in $\mathcal{F}$.

Proposition 2.1. *Let the functions $f_i: X_i \rightarrow Y_i, i = 0, 1$ belong to a family $\mathcal{F}$ which has the lifting property. Then the set*

$$D = \{((x_0, f_1(x_1)), (x_1, f_0(x_0))) : x_0 \in X_0, x_1 \in X_1\}$$

is closed, homeomorphic to $X_0 \times X_1$ such that the projections onto $X_0 \times Y_1$ and $X_1 \times Y_0$ belong to $\mathcal{F}$.

Proof. The mapping $F_i(x_i, x_{1-i}) = ((x_0, f_1(x_1)), (x_1, f_0(x_0)))$ is a homeomorphism between $X_i \times X_{1-i}$ and D , and its composition with of the projection onto $X_i \times Y_{1-i}$ is the lifting of f_{1-i} to X_i . $\square$

We call a continuous function f from a polish space to a polish space an *L function* if it maps Lusin subsets of its domain to perfectly meager sets. Lusin's function Φ is an example of a one-to-one *L function*.

Proposition 2.2. *The family of one-to-one L functions with dense in itself domains has the lifting property.*

Proof. Let $f: X \rightarrow Y$ be a one-to-one *L function*, X without isolated points. Fix a polish space Z and let S be a Lusin subset of $Z \times X$. We shall prove that the lifting $\bar{f}$ of f to Z maps S to a perfectly meager set.

First note that the projection Π along Z is a continuous open map such that the inverse images of points are meager. Therefore Π sends Lusin sets to Lusin sets. So $\Pi[S]$ is a Lusin subset of X and hence the set $f[\Pi[S]]$ must be perfectly meager.

Now let A be a fixed dense in itself subset of $\bar{f}[S]$. We want to show that A is meager in itself. Each horizontal section of A is countable because it is contained in $\bar{f}[S \cap (Z \times \{x\})]$ for some $x \in X$ and $S \cap (Z \times \{x\})$ is countable as the intersection of the Lusin set S with a meager subset of $Z \times X$. So if $\Pi[A]$ is countable, then A itself is countable, and being dense in itself must be meager in itself.

If $\Pi[A]$ is uncountable, we decompose it into two sets: A_0 —dense in itself and A_1 —countable. A_0 being a dense in itself subset of the perfectly meager set $f[\Pi[S]]$ is meager in itself and hence meager in $\Pi[A]$. Therefore $A \cap (Z \times A_0)$ is meager in A as the inverse image by an open continuous map (namely $\Pi|_A$) of a meager subset of $\Pi[A]$.

The set $A \cap (Z \times A_1)$ is also meager in A because it is a countable subset of a dense in itself set A (recall that A_1 and all horizontal sections of A are countable.)

So A is composed of two meager in A subsets and therefore A is meager in itself. $\square$

Note that the family of all L functions does not have the lifting property. A contraction of a space to one point provides a counterexample. However we can replace “one-to-one” restriction by “inverse images of points are meager.” In the proof we only have to change the point $\{x\}$ to a meager subset of X .

Proof of theorem. The theorem follows by Propositions 2.1 and 2.2 and the fact that Lusin’s function Φ is a one-to-one L function. $\square$

§3

Lusin’s function Φ has a property which makes it interesting also in the absence of Lusin sets: each subset E of the counterdomain contains a subset G comeager in E such that $\Phi^{-1}[G]$ is nowhere dense ([Mi, p. 226]). Continuous functions from polish spaces to polish spaces with this property will be called *thin*. Thin functions are L -functions ([Mi, p. 226]).

Proposition 3.1. *The family of thin functions has the lifting property.*

The proposition follows easily from the lemma given below.

Lemma. *A function $f: X \rightarrow Y$ is thin iff each basic open set $U \subseteq X$ contains an open subset V such that $f[V]$ is meager in $f[U]$.*

Proof. Only the if direction needs proving. Fix $E \subseteq Y$. Let

$$H = \bigcup \{V : V \text{ open in } X \text{ and } f[V] \text{ is meager in } E\}.$$

It is enough to show that H is open dense in X . To get this, let U be a basic open subset of X . If $f[U] \subseteq \overline{E}$, let $V \subseteq U$ be any basic open set such that $f[V]$ is meager in $f[U]$. Then $f[V]$ is also meager in E . If $f[U] \setminus \overline{E} \neq \emptyset$, let $V = f^{-1}[Y \setminus \overline{E}] \cap U$. Then V is nonempty and $f[V]$, being disjoint with E , is meager in E . So each basic open set $U \subseteq X$ contains an open subset V such that $f[V]$ is meager in E . It follows that H is an open dense set. $\square$

Lusin's function has an even stronger property than just being thin. It sends every open set to a set which is meager in itself ([K, p. 439(i)]). We shall call such functions *strongly thin*. It is clear that the family of strongly thin functions has the lifting property. It is less obvious that not all thin functions are strongly thin.

Proposition 3.2. *Let Y be a polish space which is dense in itself. There exist a polish space X and a one-to-one thin function from X onto Y .*

Proof. We can find in Y a system $\{P_\sigma : \sigma \in \omega^{<\omega}\}$ of perfect sets such that $P_\emptyset = Y$, $\sigma \leq \tau \rightarrow P_\tau \subseteq P_\sigma$, each $P_{\tau \frown i}$ is nowhere dense in P_τ and $\bigcup_i P_{\tau \frown i}$ is dense in P_τ ([S, p. 65]; [Mi, p. 226]). Let X be a polish space which is obtained from Y by declaring that all P_τ 's are open. (Given a countable family $\mathcal{A}$ of Borel sets in a polish space, we can always enrich the topology to a new polish topology in which sets from $\mathcal{A}$ are open; see [K, p. 448, the proof of Theorem 1]). We claim that the identity function from X to Y is thin. To see this note that any nonempty X -open set U must contain a nonempty set $W \cap P_\tau$ for some Y -open set W . Let i be such that the set $V = W \cap P_{\tau \frown i}$ is nonempty. We have $V \subseteq W \cap P_\tau \subseteq U$. Moreover V is an X -open set which is Y -meager in $W \cap P_\tau$. So it is Y -meager in U . $\square$

Clearly such a function as above cannot be strongly thin because its range is not meager in itself.

By combining Propositions 2.1, 3.1 and 3.2 we get the following:

Proposition 3.3. *There exist a polish space X and a Borel automorphism of X such that its graph D is a closed subset of $X \times X$ with the property that the projections restricted to D are thin functions.* $\square$

§4

Following a remark of a student of Cichoń that the range of Lusin's function is homeomorphic to $\mathbf{Q}^\omega$, we show that Lusin's function, when viewed properly, is the identity on the set $\mathbf{Q}^\omega$. This gives rise to a particularly simple proof of the basic property of Lusin's function.

Let $\mathcal{M}$ be $\mathbf{Q}^\omega$, $\mathbf{Q}$ bearing its standard topology. As $\mathcal{N}$, the Baire space of irrationals, we also take $\mathbf{Q}^\omega$, but equip $\mathbf{Q}$ with the discrete topology.

Proposition 4.1.

- (a) $\mathcal{N}$ -open sets are $\mathcal{M}$ -meager in themselves.

- (b) Any set $E \subseteq \mathbf{Q}^\omega$ contains a set $G \subseteq E$ which is $\mathcal{M}$ -comeager in E and $\mathcal{N}$ -nowheredense in $\mathbf{Q}^\omega$.
- (c) Any $\mathcal{N}$ -Lusin subset of $\mathbf{Q}^\omega$ is $\mathcal{M}$ -perfectly meager.

Proof. It is enough to show (a) for basic $\mathcal{N}$ -open sets. Let $U = \{t \in \mathbf{Q}^\omega : \tau \subseteq t\}$, where τ is a finite sequence of rationals of length n . Let $\mathbf{Q}$ have its standard topology. Then $\mathbf{Q}$ is meager in itself. Since the product of a meager in itself space with any space is meager in itself, $U = \{\tau\} \times \mathbf{Q} \times \mathbf{Q}^{\omega \setminus (n+1)}$ is meager in itself in the $\mathcal{M}$ topology. $\square$

Note that by this proposition the identity on $\mathbf{Q}^\omega$ is strongly thin when its domain and range have $\mathcal{N}$ and $\mathcal{M}$ topologies respectively. Also, when we topologize $(\mathbf{Q}^\omega)^2 \times (\mathbf{Q}^\omega)^2$ in the manner $(\mathcal{N} \times \mathcal{M}) \times (\mathcal{M} \times \mathcal{N})$, then the diagonal

$$D = \{((x, y), (x, y)) : x, y \in \mathbf{Q}^\omega\}$$

has the property that the projections act on D as strongly thin functions, i.e. send open subsets of D to sets meager in themselves.

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