ISOMETRIES ON CONSERVATIVE SUBALGEBRAS OF BOUNDED SEQUENCES

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ABSTRACT. Using extreme point techniques, we show that if A is a closed subalgebra of the bounded sequences which contain c , then any linear isometry of A onto itself is a permutation up to a modulus one multiplication. If the subalgebra A is generated by an ideal, then a permutation P maps A onto itself if and only if P maps μ -null sets to μ -null sets where μ is a $\{0,1\}$ -valued finitely additive measure associated with the ideal. In particular, if T is a nonnegative regular summability method, we characterize the isometries which map the bounded strongly T -summable sequences onto themselves and give a concrete sufficient condition for a permutation to map the bounded strongly Cesaro summable sequences onto themselves.

In this paper we establish that any surjective linear isometry of a closed subalgebra of ℓ_∞ which contains the convergent sequences can be realized as modulus one multiplication of a permutation and characterize the linear isometries for a subclass of these algebras. We also give an example of a closed subalgebra that has isometries which cannot be realized as permutations.

Banach [1] was the first to determine that the surjective linear isometries of the spaces ℓ_p , $1 \leq p < \infty$ ($p \neq 2$), were modulus one multiples of permutation maps. Since then, various authors have studied the isometries of sequence spaces possessing certain properties. For example, see [5],[6],[10],[11], and [13]. It is not surprising that the isometries found in these papers are modulus one multiples of permutations and results of Tam [11], Fleming and Jamison [6], and [7], Fleming, Goldstein, and Jamison [5] and Zaidenberg [13] allow one to deduce that the surjective linear isometries of ℓ_∞ are of this form. What is surprising is that it appears that no one has specifically addressed the space ℓ_∞ and its subalgebras, including c .

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In the following, let

$$\begin{aligned}\omega &= \text{the collection of all complex valued sequences,} \\ \varphi &= \{x \in \omega: x \text{ is finitely nonzero}\}, \\ c_0 &= \{x \in \omega: \lim x = 0\}, \\ c &= \{x \in \omega: \lim x \text{ exists}\}, \\ \ell_\infty &= \{x \in \omega: \sup_n |x_n| < \infty\},\end{aligned}$$

and let $\|x\| = \sup_n |x_n|$ whenever $x \in \ell_\infty$. We let e denote the sequence that is identically 1, e^k denote the sequence with 1 in the k th position and 0 elsewhere and, for $x \in \omega$, we write $|x| = 1$ if $|x_k| = 1$ for all $k \in \mathbf{N}$. If $\mathcal{A}$ is a subalgebra of ℓ_∞ , we say that $\mathcal{A}$ is conservative if $c \subseteq \mathcal{A}$. Note that a conservative subalgebra always contains the identity. A linear mapping T from $\mathcal{A}$ onto itself is called an isometry if $\|Tx\| = \|x\|$ for all $x \in \mathcal{A}$. Given $k \in \mathbf{N}$, we let $p_k(x) = x_k$ for all $x \in \ell_\infty$ and note that $p_k \in \mathcal{A}^*$ (the continuous dual of $\mathcal{A}$).

We also use some of the language of general matrix summability theory. If $S = (s_{n,k})$ is an infinite array of scalars where k and n range over $\mathbf{N}$ and $x, y \in \omega$, we say that $Sx = y$ if $\sum_{k=1}^\infty s_{n,k}x_k = y_n$ for all $n \in \mathbf{N}$ and call S a matrix summability method. The matrix S is called nonnegative if $s_{n,k} \geq 0$ for all $n, k \in \mathbf{N}$. A matrix summability method is called regular if $Sx \in c$ and $\lim Sx = \lim x$ whenever $x \in c$ and a sequence is said to be strongly S -summable to L if $\lim_n \sum_{k=1}^\infty s_{n,k}|x_k - L| = 0$.

1. ISOMETRIES ON CONSERVATIVE SUBALGEBRAS OF ℓ_∞

The main result of this section is that if $\mathcal{A}$ is a closed conservative subalgebra of ℓ_∞ and T is a linear isometry of $\mathcal{A}$ onto itself, then T is a permutation up to signs, i.e. $Tx = (\sigma_k x_{\pi(k)})$ where $\pi: \mathbf{N} \rightarrow \mathbf{N}$ is permutation and $|\sigma_k| = 1$ for all $k \in \mathbf{N}$. We establish this by exploiting the observations that if T is an isometry, then T^* is an isometry and that isometries carry extreme points to extreme points.

We begin by observing that if f is an extreme point of $B_{\mathcal{A}^*}$, the ball of $\mathcal{A}^*$, then there is a multiplicative linear functional $m \in \mathcal{A}^*$ and an $\alpha \in \mathbf{C}$, $|\alpha| = 1$, such that $f(x) = \alpha m(x)$ for all $x \in \mathcal{A}$. Recall that if

$$K = \left\{ m \in B_{\mathcal{A}^*}: m(xy) = m(x)m(y) \text{ and } m(\overline{x}) = \overline{m(x)}; x, y \in \mathcal{A} \right\}$$

then K is a weak-star compact subset of $\mathcal{A}^*$ and $R: \mathcal{A} \rightarrow C(K)$ defined by $(Rx)(m) = m(x)$ is a surjective linear isometry. The Arens-Kelly theorem asserts that if z^* is an extreme point of $B_{C(K)^*}$, then there is an $m \in K$ and an $\alpha \in \mathbf{C}$ such that $z^* = \alpha x_m^*$ where $|\alpha| = 1$ and $x_m^*(f) = f(m)$ for all $f \in C(K)$. Since $(R^* x_m^*)(x) = m(x)$ for all $x \in \mathcal{A}$, isometries carry extreme points to extreme points and R^* and $(R^*)^{-1}$ are both isometries, we may conclude that

any extreme point of $B_{\mathcal{A}}$ is of the form αm where $|\alpha| = 1$ and $m \in K$ [4, IV.6.18].

Theorem 1. *Let $\mathcal{A}$ be a closed subalgebra of ℓ_∞ which contains c and $T: \mathcal{A} \rightarrow \mathcal{A}$ be a surjective linear isometry. Then there exists a permutation $\pi: \mathbf{N} \rightarrow \mathbf{N}$ and a sequence $(\sigma_k) \in \mathcal{A}$, $|\sigma_k| = 1$, such that $Tx = (\sigma_k x_{\pi(k)})$ for all $x \in \mathcal{A}$; i.e., T is a permutation up to signs.*

Proof. Let $k \in \mathbf{N}$ and define $f_k \in \mathcal{A}^*$ by $f_k(x) = (Tx)_k$ for all $x \in \mathcal{A}$. Note that $\|f_k\| = 1$ and $f_k = T^* p_k$ for all $k \in \mathbf{N}$. Since p_k is an extreme point of $B_{\mathcal{A}}$ and T^* is an isometry, f_k is also an extreme point of $B_{\mathcal{A}}$ and hence $f_k = \sigma_k m_k$ where $|\sigma_k| = 1$ and m_k is a multiplicative linear functional on $\mathcal{A}$.

Now set $I_r = \{k \in \mathbf{N}: m_k(e^r) \neq 0\}$ for all $r \in \mathbf{N}$. Since $\|Te^r\| = 1$, there is a $k \in \mathbf{N}$ such that $(Te^r)_k \neq 0$ and hence $m_k(e^r) \neq 0$, i.e. $I_r \neq \emptyset$ for all $r \in \mathbf{N}$. Also note that if $r \neq s$, then $m_k(e^r e^s) = m_k(0) = 0$ and hence $I_r \cap I_s = \emptyset$. It follows that the mapping $\beta: \mathbf{N} \rightarrow \mathbf{N}$ defined by $\beta(r) = \min\{k: m_k(e^r) \neq 0\}$ is well defined and injective.

Before establishing that β is also surjective, we show that if $z \in c_0$ then $m_{\beta(r)}(z) = z_r$. Since $m_{\beta(r)}(e^r)^2$ and $m_{\beta(r)}(e^r) \neq 0$, we have that $m_{\beta(r)}(e^r) = 1$ and $m_{\beta(r)}(e^s) = 0$ whenever $s \neq r$. It follows then if $z \in \varphi$, then $m_{\beta(r)}(z) = \sum_{j=1}^{\infty} z_j m_{\beta(r)}(e^j) = z_r m_{\beta(r)}(e^r) = z_r$. Since $\overline{\varphi} = c_0$, $m_{\beta(r)}(z) = z_r$ for all $z \in c_0$.

Next we establish that $m_{\beta(r)}(x) = x_r$ for all $x \in \mathcal{A}$. Let $r \in \mathbf{N}$ be fixed and let $x \in \mathcal{A}$, $x \notin c_0$. Let $\tilde{x} = x - x_r e^r$, $z = \tilde{x}/\|\tilde{x}\|$ and $\alpha = m_{\beta(r)}(z)$. Note that $|m_{\beta(r)}(z \pm e^r)| = |\alpha \pm 1| \leq \|z \pm e^r\| \|m_{\beta(r)}\| = 1$ forces $\alpha = 0$. It follows that $m_{\beta(r)}(x) = m_{\beta(r)}(x_r e^r + \tilde{x}) = x_r$ and hence we have that $m_{\beta(r)}(x) = x_r$ for all $x \in \mathcal{A}$.

We are now prepared to show that β is surjective. Suppose that $k \in \mathbf{N} \setminus \{\beta(r): r \in \mathbf{N}\}$. Since T is surjective, there is an $x \in \mathcal{A}$ such that $Tx = e^k$. Since $(Tx)_{\beta(r)} = \sigma_r m_{\beta(r)}(x) = \sigma_r x_r$ for all $r \in \mathbf{N}$, we have that $x = 0$ and consequently $Tx = e^k = 0$. It follows that $\mathbf{N} = \{\beta(r): r \in \mathbf{N}\}$. Also note that $Te = \sigma$ forces $\sigma \in \mathcal{A}$.

Define the permutation π of $\mathbf{N}$ by $\pi = \beta^{-1}$. Now $Tx = (\sigma_k m_k(x)) = (\sigma_k m_{\beta(\pi(k))}(x)) = (\sigma_k x_{\pi(k)})$ for all $x \in \mathcal{A}$. $\square$

The preceding proof makes extensive use of the hypothesis that $c \subseteq \mathcal{A}$ in that we exploit that $\overline{\varphi} = c_0 \subseteq \mathcal{A}$ and $e \in \mathcal{A}$. There is, however, a large collection of closed subalgebras of ℓ_∞ that do not contain c . In fact, if K is an infinite compact metric space which is dense in itself then there is a closed subalgebra U of ℓ_∞ such that U is isometrically isomorphic to $C(K)$ and U does not contain c : If (x_n) is a dense sequence of distinct elements of K then U can be taken to be the image of $C(K)$ under S where $Sf = (f(x_n))$ for all $f \in C(K)$. Note that if $(f(x_n))$ is convergent then f is constant; hence U cannot contain c .

The above collection of examples indicates that the hypothesis $c \subseteq \mathcal{A}$ cannot be completely removed from the theorem. In fact, let $r = (r_n)$ be an enumeration of the rational numbers in $[0, 1]$ and let R be the linear isometry on $C[0, 1]$ defined by $Rf(x) = f(\sqrt{x})$. Define $S: C[0, 1] \rightarrow \ell_\infty$ by $Sf = (f(r_n))$. If $\mathcal{A}$ is the image of $C[0, 1]$ under S and $T: \mathcal{A} \rightarrow \mathcal{A}$ is defined by $T = SRS^{-1}$, then T is an isometry of $\mathcal{A}$ onto itself and $Tr = (\sqrt{r_n})$. Since $(\sqrt{r_n})$ contains an irrational number, it follows that T cannot be represented as a permutation up to a modulus one multiplication.

2. ISOMETRIES ON SUBALGEBRAS GENERATED BY IDEALS WHICH CONTAIN c_0

In the preceding section we established that any isometry of a conservative subalgebra onto itself can be realized as a permutation. The converse, however, does not hold for arbitrary conservative subalgebras. In order to see this, recall that if x and y are both divergent sequences of 0's and 1's then there is a permutation π such that $y_k = (x_{\pi(k)})$. It follows immediately that if $\mathcal{A}$ is a subalgebra such that there are divergent sequences of 0's and 1's such that $x \in \mathcal{A}$ but $y \notin \mathcal{A}$, then there is a permutation P that does not map $\mathcal{A}$ onto itself. We will give an example of a collection of such conservative subalgebras later in this section.

In this section we characterize the isometries on proper subalgebras "generated by ideals which contain c_0 " which are constructed as follows: Let M be a proper closed ideal of ℓ_∞ such that $c_0 \subseteq M$ and set

$$\mathcal{A}_M = \{x \in \ell_\infty : x - Le \in m \text{ for some } L \in \mathbb{C}\}.$$

It is easy to verify that $\mathcal{A}_M$ is a closed subalgebra of ℓ_∞ , and since $c_0 \subseteq M, c \subseteq \mathcal{A}_M$. Also, note that if $x - Le \in M$ then x has a subsequence which is convergent to L (otherwise $M = \ell_\infty$). Throughout the sequel, we always let M denote a closed ideal of ℓ_∞ such that $c_0 \subseteq M$ and, if $x \in \ell_\infty$ and $\varepsilon > 0$, $A(\varepsilon, x) = \{k \in \mathbb{N} : |x_k| \geq \varepsilon\}$. If $L \in \mathbb{C}$ we let $x - L$ denote the sequence $x - Le$.

It is convenient to describe subalgebras using finitely additive two-valued measures. Given M , set $\Gamma_M = \{A \subseteq \mathbb{N} : \chi_A \text{ or } \chi_{A^c} \in M\}$ and define $\mu_m: \Gamma_M \rightarrow \{0, 1\}$ by $\mu_M(A) = 0$ if $\chi_A \in M$ and $\mu_M(A) = 1$ if $\chi_{A^c} \in M$. It is straightforward to check that Γ_M is a subfield of $\mathcal{P}(\mathbb{N})$, the power set of $\mathbb{N}$, and that μ_M is a finitely additive measure. Also note that if $B \in \Gamma_M, \mu_M(B) = 0$ and $A \subseteq B$, then $\mu_M(A) = 0$; i.e., μ_M is complete [3].

The following lemma provides a useful test for membership in a closed ideal of ℓ_∞ .

Lemma 2. *Let $x \in \ell_\infty$ and M be a closed ideal of ℓ_∞ . Then $x \in M$ if and only if $\mu_M(A(\varepsilon, x)) = 0$ for all $\varepsilon > 0$.*

Proof. Observe that $\chi_{A(\varepsilon, x)} = xy$ if $y \in \ell_\infty$ is defined by $y_k = 1/x_k$ when $|x_k| \geq \varepsilon$ and 0 otherwise. Since M is an ideal, it follows that if $x \in M$ then $\chi_{A(\varepsilon, x)} \in M$ for all $\varepsilon > 0$.

Now note that $\|x - x\chi_{A(\varepsilon, x)}\| < \varepsilon$ for all $\varepsilon > 0$. Since M is closed, we have that $x \in M$ if $\chi_{A(\varepsilon, x)} \in M$ for all $\varepsilon > 0$. $\square$

A consequence of this lemma is that $x \in \mathcal{A}_M$ if and only if

$$\mu_M(A(\varepsilon, x - L)) = 0$$

for all $\varepsilon > 0$ for some $L \in \mathbb{C}$.

There is an abundant collection of such subalgebras. If $S = (s_{n,k})$ is a nonnegative regular summability method, then

$$M(S) = \left\{ x \in \ell_\infty : \lim_n \sum_{k=1}^{\infty} s_{n,k} |x_k| = 0 \right\}$$

is a closed ideal of ℓ_∞ [3], [8] and the subalgebra $\mathcal{A}_{M(S)}$ consists of the bounded strongly S -summable sequence. Since S is regular, Steinhaus's theorem on summability [12] yields that there is a sequence of 0's and 1's that is not strongly S -summable and hence $\mathcal{A}_{M(S)} \neq \ell_\infty$. We also note that

$$\Gamma_{M(S)} = \left\{ A \subset \mathbb{N} : \lim_n \sum_{k=1}^{\infty} s_{n,k} \chi_A(k) = 0 \text{ or } 1 \right\}$$

and $\mu_{M(S)}(A) = \lim_n \sum_{k=1}^{\infty} s_{n,k} \chi_A(k)$ for all $A \in \Gamma_{M(S)}$. It can also be shown that $\mathcal{A}_{M(S)}$ is the collection of bounded strongly S -summable sequences [3].

Theorem 3. Let $\mathcal{A}_M$ be the subalgebra generated by M and suppose $\mathcal{A}_M \neq \ell_\infty$. Then T is a linear isometry $\mathcal{A}_M$ onto itself if and only if $Tx = (\sigma_k x_{\pi(k)})$ where

- (1) $(\sigma_k) \in \mathcal{A}_M$ and $|\sigma_k| = 1$ for all $k \in \mathbb{N}$;
- (2) π is a permutation of $\mathbb{N}$ such that $\pi(A) \in \Gamma_M$ if and only if $A \in \Gamma_M$;
- (3) $\mu_M(A) = 0$ if and only if $\mu_M(\pi(A)) = 0$ for all $A \in \Gamma_M$.

Proof. We let $Px = (x_{\pi(k)})$ and $M_\sigma(x) = (\sigma_k x_k)$ for all $x \in \ell_\infty$, and we let $\mu = \mu_M$, $\mathcal{A} = \mathcal{A}_M$ and $\Gamma = \Gamma_M$.

Sufficiency. We establish that M_σ and P are both linear isometries of $\mathcal{A}$ onto itself and hence $T = M_\sigma P$ is a surjective linear isometry.

First note that, since $|\sigma_k| = 1$ for all k , $\|M_\sigma x\| = \|x\|$ for all $x \in \mathcal{A}$ and the sequence $\sigma^{-1} = (\sigma_k^{-1}) \in \ell_\infty$. Since $\sigma \in \mathcal{A}$, $M_\sigma x \in \mathcal{A}$ for all $x \in \mathcal{A}$ and there is an $L \in \mathbb{C}$ such that $\sigma - L \in M$. Since some subsequence of σ must converge to L , $|L| = 1$ and L^{-1} exists. Now $-L^{-1}\sigma^{-1}(\sigma - Le) = \sigma^{-1} - L^{-1} \in M$ and hence $\sigma^{-1} \in \mathcal{A}$. It follows immediately that M_σ is an isometry of $\mathcal{A}$ onto itself.

Next we establish that P is a linear isometry of $\mathcal{A}$ onto itself. First we establish that P maps M into M . Let $x \in M$ and observe that

$$\begin{aligned} A(\varepsilon, Px) &= \{r : |(Px)_r| \geq \varepsilon\} = \{r : |x_{\pi(r)}| \geq \varepsilon\} \\ &= \{\pi^{-1}(s) : |x_s| \geq \varepsilon\} = \pi^{-1}(A(\varepsilon, x)). \end{aligned}$$

Since $\mu(A(\varepsilon, x)) = 0$, $\mu(\pi^{-1}(A(\varepsilon, x))) = \mu(A(\varepsilon, Px)) = 0$ for all $\varepsilon > 0$ and hence $Px \in M$. If $y \in \mathcal{A}$, then $y - Le \in M$ for some L and hence $P(y - Le) = Py - Le \in M$, i.e., $P\mu \in \mathcal{A}$.

Since the same argument shows that P^{-1} maps $\mathcal{A}$ into $\mathcal{A}$ and it is clear that P is a linear isometry, we have that P is a surjective linear isometry.

Necessity. Now suppose that $T: \mathcal{A} \rightarrow \mathcal{A}$ is a surjective linear isometry. We have already established that there is a permutation π and $\sigma \in \mathcal{A}$, $|\sigma_k| = 1$ for all k , such that $T = M_\sigma P$. The proof will be completed once we have established (2) and (3).

The preceding part of the proof establishes that $\sigma^{-1} \in \mathcal{A}$. It follows that $P = M_{\sigma^{-1}} M_\sigma P$, and hence P^{-1} , are linear isometries of $\mathcal{A}$ onto itself. Also, if $A \in \Gamma$ then $P(\chi_A)$ and $P^{-1}(\chi_A)$ are both in $\mathcal{A}$. Since $P^{-1}(\chi_A) = \chi_{\pi(A)}$, and hence either $\chi_{\pi(A)}$ or $e - \chi_{\pi(A)} = \chi_{\pi(A)^c}$ must be in M , we have that $\pi(A) \in \Gamma$ if $A \in \Gamma$. Similarly, since $P(\chi_A) = \chi_{\pi^{-1}(A)}$, we have that $A \in \Gamma$ if $\pi(A) \in \Gamma$.

Now suppose that $A \in \Gamma$, $\mu(A) = 0$ and $\mu(\pi(A)) = 1$. Since $\mathcal{A} \neq \ell_\infty$, there is a $D \subseteq \mathbb{N}$ such that $\chi_D \notin \mathcal{A}$ [2]. Write $A \cup D = A \cup F$ where $A \cap F = \emptyset$ and note that $F \notin \Gamma$. Since $\pi(F) \subseteq \pi(A)^c$ and $\mu(\pi(A)^c) = 0$, the completeness of μ forces $F \in \Gamma$. The necessity of (3) follows and hence the theorem. $\square$

As the next example indicates, the hypothesis $\mathcal{A}_M \neq \ell_\infty$ cannot be eliminated from the preceding theorem. Mazur [9] has established the existence of a linear functional $f \in \ell_\infty^*$ such that $f(xy) = f(x)f(y)$ for all $x, y \in \ell_\infty$ and $f(x) = \lim x$ for all $x \in c$. Observe that, for any $A \subseteq \mathbb{N}$, $f(\chi_A) = f(\chi_A)^2$ and hence $f(\chi_A) = 0$ or $f(\chi_A) = 1$. If we set $M = \{x \in \ell_\infty: f(x) = 0\}$, M is a closed ideal, $\Gamma_M = P(\mathbb{N})$ and $\mu_M(A) = f(\chi_A)$. Also note that if $x \in \ell_\infty$ and $f(x) = L$, then $x - L \in M$ and hence $\mathcal{A}_M = \ell_\infty$. Now let $A \subseteq \mathbb{N}$ such that $\chi_A \notin c$ and π be a permutation such that $\pi(A) = A^c$. Since any permutation determines an isometry of ℓ_∞ onto itself, π , determines an isometry of $\mathcal{A}_M$ onto itself that does not take μ_M -null sets to μ_M -null sets.

We can use the above theorem to give a sufficient condition for an isometry to map the bounded strongly Cesaro sequences onto themselves. First recall that a sequence $x = (x_k)$ is strongly Cesaro summable to L provided $\lim_n n^{-1} \sum_{k=1}^n |x_k - L| = 0$. It can be established that a bounded sequence is strongly Cesaro summable to L if and only if $\lim_n n^{-1} \sum_{k=1}^n \chi_{A(\varepsilon, x-L)}(k) = 0$ for all $\varepsilon > 0$ [3]. Let S denote the collection of bounded sequences which are strongly Cesaro summable to 0. It is now straightforward to verify that S is an ideal which, via Lemma 2, is closed in ℓ_∞ . Also note that, by Steinhaus's theorem on summability $A_S \neq \ell_\infty$.

Definition 4. We call a permutation π of $\mathbb{N}$ a block permutation if there is a sequence of integers $1 = \nu(1) < \nu(2) < \dots$ and permutations

$$\pi_k: [\nu(k), \nu(k+1)) \rightarrow [\nu(k), \nu(k+1))$$

such that $\pi_k(r) = \pi(r)$ for all $r \in [\nu(k), \nu(k+1))$.

Proposition 5. Suppose π is a block permutation of $\mathbf{N}$ as described in the preceding definition. If $\limsup_k \nu(k+1)/\nu(k) = 1$, then the mapping $Px = (x_{\pi(k)})$ maps the bounded strongly Cesaro summable sequences onto themselves.

Proof. A straightforward computation shows that $\chi_A \in S$ if and only if $\chi_A \in S$ which, by the preceding theorem, establishes the proposition.

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