FIRST EIGENVALUE OF THE LAPLACEAN AND TORSION OF PARALLELIZABLE RIEMANNIAN MANIFOLDS

PATRICK GHANAAT

(Communicated by Jonathan M. Rosenberg)

ABSTRACT. Lower and upper bounds for the smallest positive eigenvalue of the Laplacean on a parallelizable Riemannian manifold are combined to obtain an explicit lower bound for the torsion of global orthonormal frame fields in terms of the diameter of the metric.

INTRODUCTION AND RESULTS

Consider a compact, connected n -dimensional Riemannian manifold M with metric tensor g . We assume that the tangent bundle TM is trivial. Then there exist parallelizations $\omega: TM \rightarrow \mathbf{R}^n$ mapping g into the euclidean metric on $\mathbf{R}^n$. Such ω will be called orthonormal. Equivalently, the vector fields E_i defined by $\omega(E_i) = e_i = i$ th standard basis vector of $\mathbf{R}^n$ form an orthonormal frame field on M .

The torsion T of ω is defined by $T(X, Y) = \omega^{-1}(d\omega(X, Y))$ for vector fields X and Y on M . T is the torsion tensor of the flat connection D on TM defined by $DE_i = 0$. It is well known that a compact Riemannian manifold admits a torsion-free orthonormal parallelization if and only if it is isometric to a flat torus.

In this note we study the relation between isometric invariants of a Riemannian manifold and the torsion of its orthonormal frame fields. Specifically, let κ denote the maximum norm of T , i.e.,

$$\kappa = \max\{\|T(X, Y)\| : X, Y \in T_x M, \|X\| = \|Y\| = 1, x \in M\}.$$

The smallest positive eigenvalue of the Laplace operator of (M, g) on functions is denoted by λ_1 .

Theorem 1. *Let (M, g) be a compact connected Riemannian manifold and $\omega: TM \rightarrow \mathbf{R}^n$ an orthonormal parallelization. If for some p ($1 \leq p \leq n-1$) the Betti number b_p of M satisfies $b_p > \binom{n}{p}$, then $\lambda_1 \leq 100 n^8 \kappa^2$.*

Example. Let $M = \Gamma \backslash G$ be a nilmanifold. Then there exist a sequence of Riemannian metrics $\{g_\nu\}$ and a corresponding sequence $\{\omega_\nu\}$ of orthonormal

Received by the editors April 10, 1989.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 53C20.

parallelizations on M such that κ^2/λ_1 converges to zero as $\nu \rightarrow \infty$. For g_ν one can use the almost flat metrics constructed by Gromov and for ω_ν Maurer-Cartan forms (see [2]).

Theorem 1 can be combined with lower bounds on λ_1 . Here we note the following consequence:

Theorem 2. *Under the assumptions of Theorem 1 we have $\kappa d \geq 50^{-n}$, where d is the diameter of (M, g) .*

The existence of some positive $\varepsilon = \varepsilon(n)$ such that $\kappa d \geq \varepsilon$ also follows from [1].

THE PROOF

We use tensor notation and the summation convention for the computations. Components of tensors are taken with respect to the orthonormal frame $\{E_i\}$ and coframe $\{\omega^i\}$, where $\omega = \omega^i \otimes e_i$. For example, $T = T_{ij}^k \omega^i \otimes \omega^j \otimes E_k$. Since $g_{ij} = \delta_{ij}$, we need not distinguish between upper and lower indices.

If ∇ denotes the Levi-Civita connection of g then $\nabla - D = S$, where S is given by $S_{ijk} = \frac{1}{2}(-T_{ij}^k + T_{jk}^i - T_{ki}^j)$. We obtain the following expressions for the exterior derivative d and its formal adjoint δ acting on a p -form α :

$$(d\alpha)_{i_0 \dots i_p} = \sum_{\nu=0}^p (-1)^\nu \alpha_{i_0 \dots \widehat{i_\nu} \dots i_p, i_\nu} + \sum_{\mu < \nu} (-1)^{\mu+\nu} T_{i_\mu i_\nu}^j \alpha_{j i_0 \dots \widehat{i_\mu} \dots \widehat{i_\nu} \dots i_p}$$

$$(\delta\alpha)_{i_2 \dots i_p} = -\alpha_{ii_2 \dots i_p, i} + T_{ik}^i \alpha_{ki_2 \dots i_p} + \sum_{\nu=2}^p S_{ii_\nu k} \alpha_{ii_2 \dots i_{\nu-1} k i_{\nu+1} \dots i_p}.$$

Here $\alpha_{i_0 \dots \widehat{i_\nu} \dots i_p, i_\nu} = E_{i_\nu}(\alpha_{i_0 \dots \widehat{i_\nu} \dots i_p}) = (D\alpha)_{i_0 \dots \widehat{i_\nu} \dots i_p i_\nu}$. In particular, Green's Theorem for a vector field $X = X_i E_i$ can be written

$$\int_M X_{i,i} dV = - \int_M T_{ki}^i X_k dV$$

with $dV = \omega^1 \wedge \dots \wedge \omega^n$.

Lemma 1. *For vector fields X and Y on M ,*

$$\begin{aligned} \int_M X_{i,j} Y_{j,i} dV &= \int_M X_{i,i} Y_{j,j} dV \\ &\quad + \int_M (T_{jk}^k X_{i,i} - T_{ik}^k X_{i,j} + T_{ij}^k X_{i,k}) Y_j dV. \end{aligned}$$

Proof. This follows from Green's Theorem and $[E_i, E_j] = -T_{ij}^k E_k$.

Lemma 2. *For any harmonic p -form α , the L^2 -norm of the covariant derivative $D\alpha$ satisfies $\|D\alpha\|_2 \leq 10 n^4 \kappa \|\alpha\|_2$.*

Proof. We compute, using the summation convention.

$$\begin{aligned} \|d\alpha\|_2^2 &= \sum_{\mu=0}^p \int_M \alpha_{i_0 \dots \widehat{i_\mu} \dots i_p, i_\mu} \alpha_{i_0 \dots \widehat{i_\mu} \dots i_p, i_\mu} dV \\ &\quad + \sum_{\mu \neq \nu} (-1)^{\mu+\nu} \int_M \alpha_{i_0 \dots \widehat{i_\mu} \dots i_p, i_\mu} \alpha_{i_0 \dots \widehat{i_\nu} \dots i_p, i_\nu} dV \\ &\quad + 2 \sum_{\substack{\rho=0 \\ \mu < \nu}}^p (-1)^{\rho+\mu+\nu} \int_M \alpha_{i_0 \dots \widehat{i_\rho} \dots i_p, i_\rho} T_{i_\mu i_\nu}^j \alpha_{j i_0 \dots \widehat{i_\mu} \dots \widehat{i_\nu} \dots i_p} dV \\ &\quad + \sum_{\substack{\mu < \nu \\ \rho < \sigma}} (-1)^{\mu+\nu+\rho+\sigma} \int_M T_{i_\mu i_\nu}^j T_{i_\rho i_\sigma}^k \alpha_{j i_0 \dots \widehat{i_\mu} \dots \widehat{i_\nu} \dots i_p} \alpha_{k i_0 \dots \widehat{i_\rho} \dots \widehat{i_\sigma} \dots i_p} dV. \end{aligned}$$

The first sum is equal to $(p+1)\|D\alpha\|_2^2$. The absolute values of the third and fourth sums are bounded above by $(p+1)n^4\kappa\|\alpha\|_2\|D\alpha\|_2$ and by $(p+1)n^5\kappa^2\|\alpha\|_2^2$, respectively. We apply Lemma 1 and the fact that $\delta\alpha = 0$ to obtain an upper bound $6(p+1)n^4\kappa\|\alpha\|_2\|D\alpha\|_2$ for the absolute value of the second sum: By Lemma 1,

$$\begin{aligned} \int_M \alpha_{i_0 \dots \widehat{i_\mu} \dots i_p, i_\mu} \alpha_{i_0 \dots \widehat{i_\nu} \dots i_p, i_\nu} dV &= \int_M \alpha_{i_0 \dots \widehat{i_\mu} \dots i_p, i_\nu} \alpha_{i_0 \dots \widehat{i_\nu} \dots i_p, i_\mu} dV \\ &\quad + \int_M (T_{i_\mu k}^k \alpha_{i_0 \dots \widehat{i_\mu} \dots i_p, i_\nu} - T_{i_\nu k}^k \alpha_{i_0 \dots \widehat{i_\mu} \dots i_p, i_\mu} \\ &\quad + T_{i_\nu i_\mu}^k \alpha_{i_0 \dots \widehat{i_\mu} \dots i_p, k}) \alpha_{i_0 \dots \widehat{i_\nu} \dots i_p} dV. \end{aligned}$$

Since $\delta\alpha = 0$ and $\mu \neq \nu$, we have

$$\alpha_{i_0 \dots \widehat{i_\mu} \dots i_p, i_\nu} = \pm T_{ik}^i \alpha_{k i_0 \dots \widehat{i_\mu} \dots \widehat{i_\nu} \dots i_p} \pm \sum_{\substack{\rho=0 \\ \rho \neq \mu, \nu}}^p S_{i_\rho k} \alpha_{i i_0 \dots \widehat{i_\mu} \dots \widehat{i_\nu} \dots i_{\rho-1} k i_{\rho+1} \dots i_p}.$$

After substituting these expressions, a straightforward estimate yields the required bound on the second sum. Since $d\alpha = 0$, we obtain

$$\|D\alpha\|_2^2 \leq 7n^4\kappa\|\alpha\|_2\|D\alpha\|_2 + n^5\kappa^2\|\alpha\|_2^2.$$

Lemma 2 follows by completing the square.

Let $K_\omega \subset L^2(\Lambda^p T^*M)$ denote the space of parallel p -forms with respect to D and $K_\omega^\perp$ its orthogonal complement in L^2 . K_ω is the $\mathbf{R}$ -span of $\{\omega^{i_1} \wedge \dots \wedge \omega^{i_p} : 1 \leq i_1 < i_2 < \dots < i_p \leq n\}$ and therefore has dimension $\binom{n}{p}$.

Lemma 3. For any smooth p -form $\alpha \in K_\omega^\perp$, $\sqrt{\lambda_1}\|\alpha\|_2 \leq \|D\alpha\|_2$.

Proof. Recall that $\lambda_1 = \inf \left(\|df\|_2^2 / \|f\|_2^2 \right)$, where the infimum is taken over all functions $f \in C^\infty(M)$ such that $\int_M f dV = 0$. We also note that $\alpha \in K_\omega^\perp$

if and only if each component function of α is orthogonal to the constants in L^2 . Thus,

$$\|\alpha\|_2^2 = \sum_{i_1, \dots, i_p} \|\alpha_{i_1 \dots i_p}\|_2^2 \leq \lambda_1^{-1} \sum_{i_1, \dots, i_p, j} \|\alpha_{i_1 \dots i_p, j}\|_2^2 = \lambda_1^{-1} \|D\alpha\|_2^2.$$

The hypothesis $b_p > \binom{n}{p}$ implies that there exists a nonzero harmonic form in $K_\omega^\perp$. Lemmas 2 and 3 now imply Theorem 1. To prove Theorem 2 we recall [1, Theorem 3.1]: If $\kappa d \leq 0.05$ then $\lambda_1 \geq 20^{-n} d^{-2} \lambda_1^*$, where λ_1^* is the smallest positive eigenvalue of the Laplacean on the n -dimensional euclidean unit ball under Neumann boundary conditions. By a result of Payne and Weinberger [3], $\lambda_1^* \geq \pi^2/4$, and Theorem 2 follows from Theorem 1.

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MATHEMATISCHES INSTITUT DER UNIVERSITÄT, RHEINSPRUNG 21, CH-4051 BASEL, SWITZERLAND