CYCLIC SURGERY ON KNOTS

SHICHENG WANG

(Communicated by Frederick R. Cohen)

ABSTRACT. We get a necessary condition under which a nonsimple knot (i.e. a satellite knot) admits a nontrivial surgery producing a lens space. Interesting corollaries are: (1) if a lens space can be obtained from a nontrivial surgery on a nonsimple knot, then the order of its fundamental group is not smaller than 23; (2) any nonsimple knot admits at most one nontrivial surgery which produces a lens space.

Some topologists are interested in the problem of doing surgery on a knot complement to get a lens space. This note is devoted to providing more information on this problem. Our main result is the following:

Theorem 1. *If a lens space can be obtained from a nontrivial surgery on a nonsimple knot (i.e. a satellite knot) $J(K)$, then K is a torus knot and J is a closed braid in a tubular neighborhood of K .*

There are two interesting corollaries of Theorem 1.

D. Gabai proved that one cannot get a lens space with fundamental group of order 1 (i.e. S^3) from a nontrivial surgery on a nonsimple knot (in fact nonsimple knots have property P). J. Bailey and D. Rolfsen [BR] (also R. Fintushel and R. Stern [FS] and C. Gordon [G]) proved that some lens space with fundamental group of order 23 can be obtained from a nontrivial surgery on a nonsimple knot. The first corollary says that 23 is the smallest possible.

Corollary 1. *If a lens space can be obtained from a nontrivial surgery on a nonsimple knot (i.e. a satellite knot), then the order of its fundamental group is not smaller than 23.*

M. Culler, C. M. Gordon, J. Luecke, and P. Shalen [CGLS] proved that there are at most two nontrivial surgeries on a nontorus knot which produce lens spaces. Corollary 2 below says that a nontorus knot which admits two nontrivial surgeries producing lens spaces must be a hyperbolic knot. (Berge found infinitely many such nontorus knots.)

Received by the editors February 4, 1988, and in revised form, August 17, 1988.
1980 *Mathematics Subject Classification* (1985 Revision). Primary 57N10.

© 1989 American Mathematical Society
0002-9939/89 \$1.00 + \$.25 per page

Corollary 2. *Any nonsimple knot admits at most one nontrivial surgery which produces a lens space.*

Let K be a nontrivial knot in S^3 , $N(*)$ denote tubular neighborhood. $X = E(K) = S^3 - \text{int } N(K)$.

Let $f: S^1 \times D^2 \rightarrow N(K)$ be a homeomorphism which determines the 0-frame of K , i.e. $f(S^1 \times *) = 0 \in H_1(X)$. Now choose an arbitrary orientation of K . Let $\lambda = f(S^1 \times *)$ be the longitude and $\mu = f(* \times \partial D^2)$ be the meridian. If $r = \frac{p}{n} \in Q \cup \infty$, $p, n \in \mathbb{Z}$, and $(p, n) = 1$, then $(E(K), r) = (K, r)$, and the result of Dehn surgery of type r on K is $X \cup_h S^1 \times D^2$, where $h: S^1 \times \partial D^2 \rightarrow \partial X$ is a homeomorphism such that $h(* \times \partial D^2) = p\mu + n\lambda \in H_1(\partial X)$.

S denotes a solid torus. J will always denote a circle in S which does not lie in a ball in S and is not isotopic to the center line of S . The winding number w of J is the integer $w \geq 0$ such that $H_1(J) \rightarrow H_1(S)$ has image wZ . The exterior of J is $Y = S - \text{int } N(J)$.

Let K be a knot and $f: S \rightarrow N(K)$ as above. Then $J(K)$ denotes the knot $f(J)$ in S^3 . We call $J(K)$ a nonsimple knot or a satellite knot.

Let $\alpha, \beta \in H_1(S^1 \times \partial D^2)$ be given by $\alpha = S^1 \times *$, $\beta = * \times \partial D^2$. Note that $H_1(Y) = Z\alpha \oplus Z\mu_J$, where μ_J is a meridian of $N(J)$. There exists a homeomorphism $g: S^1 \times D^2 \rightarrow N(J)$ such that $g(S^1 \times *) = w\alpha$ in $H_1(Y)$. Then $\lambda_J = g(S^1 \times *) \in H_1(\partial N(J))$ is a longitude of J . For $r = \frac{p}{n}$, we define (J, r) to be $Y \cup_h S^1 \times D^2$, where $h: S^1 \times \partial D^2 \rightarrow \partial N(J)$ is a homeomorphism such that $h(* \times \partial D^2) = p\mu_J + n\lambda_J \in H_1(\partial N(J))$. Note that for every K , if f, g are as above, then fg determines the 0-frame of $J(K)$. Hence $(J(K), r) = X \cup (J, r)$, identified via the restriction of f to $\partial(J, r) = \partial S \rightarrow \partial N(K) = \partial X$.

$Z_n = \frac{Z}{nZ} \cdot H_1(*) = H_1(*, Z)$. $L(p)$ is some lens space $L(p, n)$. T is the torus. $T_{p,q}$ is the (p, q) torus knot.

The proof of Theorem 1 will employ the following results:

[CGLS]: If K is not a torus knot then $\pi_1(K, r)$ can be cyclic only if r is integer. Moreover, there are at most two such integers r , and if there are two then they must be successive. Finally, if $\pi_1(K, r) = 0$, then $r = 1$ or -1 .

[Ga2] (see also [S]): Let M be an irreducible cobordism between tori T_0 and T_1 without nonboundary parallel essential torus, F be a Thurston norm minimizing surface representing an element in $H_2(M, T_1)$ which does not hit T_0 . Then for all but at most one slope r on T_0 , (M, r) is irreducible and F is still Thurston norm minimizing in (M, r) , where (M, r) is constructed by attaching a solid torus S to T_0 such that r bounds a disk in S .

[Ga1]: Let J be a knot in $S^1 \times D^2$ whose geometric intersection with $* \times D^2$ is nonzero. If M is a manifold obtained by a nontrivial surgery on J and M is a solid torus, then J is a braid in $S^1 \times D^2$. Satellite knots have property

P , i.e., nontrivial surgery on a satellite knot yields a nonsimply connected manifold.

[G]: (1) $H_1(J, r) = Z \oplus Z_{(w, p)}$.

(2) If $(J(K), r)$ is a lens space $L(p)$ for some integer p , then $(J, r) \cong S\#L(q)$, where q is 1 or p , and $(J(K); r) = (K; \frac{p}{w^2})\#L(p)$.

Proof of Theorem 1. For a nonsimple knot $\bar{K} = J(K)$, Suppose $(J(K); r)$ is a lens space. By [CGLS], r is an integer p . By [G], we have

$$L(p) = (J(K), p) = (K; \frac{p}{w^2})\#L(q)$$

and

$$(J; p) = S\#L(q).$$

If $q = 1$, then $(J; p) = S$. Since J does not lie in a ball in $N(K)$, its geometric intersection number with a meridian disk of $N(K)$ is nonzero. As the surgery is nontrivial, by [Ga1], J is a braid in $N(K)$; therefore $w \geq 2$ (otherwise J is a core of $N(K)$). Next we claim that K is a torus knot, because $w \geq 2$, $\frac{p}{w^2} \neq p$. If $\frac{p}{w^2}$ is an integer, then

$$(K; \frac{p}{w^2}) \neq L(p)$$

since $H_1(K, \frac{p}{w^2}) = Z \frac{p}{w^2} \neq Z_p$. So $\frac{p}{w^2}$ is not an integer. By [CGLS], K must be a torus knot. So in the case $q = 1$, we are done.

If $q = p$, then $(K; \frac{p}{w^2}) = S^3$.

Base on [Ga2], applying the same argument of the proof of Corollary 5.2 of [S] to the situation $(J, p) = S\#L(p)$, one can show that w is not zero.

Furthermore, K is not a torus knot (torus knots have property P) which implies that $p = w^2$ or $-w^2$ by [CGLS]. Now we have $H_1(J, \pm w^2) = Z \oplus Z_{(w, w^2)} = Z \oplus Z_w$ and $(J, \pm w^2) = S\#L(w^2)$ by [G]. This implies that w must be ± 1 . But then $p = \pm 1$, so $(\bar{K}, r) = (\bar{K}, \pm 1) = S^3$, which contradicts the fact that satellite knots have Property P .

Theorem 1 is proved. $\square$

The proof of two corollaries also need the following result of L. Moser (see [M]).

$$\begin{aligned} (T_{s,t}; \frac{m}{n}) &= L(s)\#L(t) \quad \text{if } \frac{m}{n} = st. \\ (T_{s,t}; \frac{m}{n}) &= L(m) \quad \text{if } m = nst \pm 1. \end{aligned}$$

Otherwise $(T_{s,t}; \frac{m}{n})$ is a Seifert manifold with orbifold S^2 on which there are three branch points of order s , $|t|$, and $|nst - m|$.

Proof of Corollary 1. Let $\bar{K}$ be a satellite knot and $(\bar{K}; r) = L(p)$.

By [CGLS], $r = p$. By Theorem 1, $\overline{K} = J(T_{s;t})$, where J is a braid in a tubular neighborhood of the torus knot $T_{s;t}$ with winding number $w \geq 2$. So now we have

$$L(p) = (J(T_{s;t}); p) = (T_{s;t}, \frac{p}{w^2}) \# L(q).$$

Torus knots have property P , hence

$$(T_{s;t}, \frac{p}{w^2}) \neq S^3.$$

Therefore $q = 1$, and we have

$$L(p) = (T_{s;t}, \frac{p}{w^2}).$$

p and w must be coprime. By [M], $p = w^2 st \pm 1$. The minimum st such that $T_{s;t}$ is nontrivial is 2×3 . So the minimum $p = w^2 st \pm 1$ is $2^2 \times 2 \times 3 - 1 = 23$. $\square$

Proof of Corollary 2. Let K be a knot such that $(K; n)$ and $(K; p)$ are lens spaces. If K is a satellite knot, from the proof of Theorem 1, we see that n, p are integers which satisfy $n = w^2 st \pm 1$, and $p = w^2 st \pm 1$. (Note that w, s, t , are determined by K). Obviously, for any choice of $w, s, t; p \neq n \pm 1$. By [CGLS], this is impossible. So K is a simple knot. $\square$

ACKNOWLEDGMENTS

I would like to express my gratitude to professor R. D. Edwards, Q. Zhou, and Y. Wu for their help. I also wish to thank the referee for his comments and suggestions.

REFERENCES

- [BR] J. Bailey and D. Rolfsen, *An unexpected surgery construction of lens space*, Pacific J. Math. **71** (1977) 295–298.
- [CGLS] M. Culler, C. M. Gordon, J. Luecke, and P. B. Shalen, *Dehn surgery on knots*, Ann. of Math. (2) **125** (1987), 237–300.
- [FS] R. Fintushel and R. Stern, *Constructing lens spaces by surgery on knots*, Math. Z. **175** (1980), 33–51.
- [Ga1] D. Gabai, *Foliations and surgery on knots*, Bull. Amer. Math. Soc. **15** (1986), 83–86.
- [Ga2] —, *Foliations and the topology of 3-manifolds II*, J. Differential Geom. **26** (1987) 461–478.
- [G] C. M. Gordon, *Dehn surgery and satellite knots*, Trans. Amer. Math. Soc. **275** (1983), 687–708.
- [M] L. Moser, *Elementary surgery along a torus knot*, Pacific J. Math. **38** (1971) 734–745.
- [S] M. Scharlemann, *Sutured manifolds and generalized Thurston norms*, J. Differential Geom. (to appear).

MATHEMATICS DEPARTMENT, UNIVERSITY OF CALIFORNIA, BERKELEY, BERKELEY, CALIFORNIA, 94720