SELF-ADJOINTNESS OF THE MOMENTUM OPERATOR WITH A SINGULAR TERM

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. Self-adjointness is shown of the momentum operator $p = -i(d/dx) + i(c/x)R$, $Ru(x) = u(-x)$, with domain $\{u \in H^1(\mathbf{R}^1): u/x \in L^2(\mathbf{R}^1)\}$ when $c > 1$ or $c < -1$. This operator appears in a harmonic oscillator system with the generalized commutation relations by Wigner: $ip = [x, H]$ and $-ix = [p, H]$ for the Hamiltonian H and the multiplication operator x .

The proof is carried out by generation of a unitary group in terms of ip , based on the Hille-Yosida theorem and Stone's theorem. The result is applied to the self-adjointness of $H = (p^2 + x^2)/2$.

1. INTRODUCTION AND THE MAIN RESULT

This paper concerns the self-adjointness of the operator:

$$p = -i(d/dx) + i(c/x)R, \quad Ru(x) = u(-x),$$

with a real constant c specified later. This operator, called the momentum operator, was expressed in this form by Yang [6] to deal with a one-dimensional harmonic oscillator system governed by the generalized commutation relations, due to Wigner:

$$ip = [x, H] \quad \text{and} \quad -ix = [p, H].$$

Here the Hamiltonian is of the classical form $H = (p^2 + x^2)/2$, and x is assumed to be the multiplication by x .

In the case of $c = 0$, p is a well-known example of a self-adjoint operator in $L^2(\mathbf{R}^1)$ (see e.g. Yosida [7, p. 198]). The purpose of this paper is to prove the following theorem, and then to apply it to the self-adjointness of the Hamiltonian H :

Theorem 1. *If $|c| > 1$, then the operator p is self-adjoint on the set*

$$V = \{u \in H^1(\mathbf{R}^1): u/x \in L^2(\mathbf{R}^1)\}.$$

Received by the editors January 30, 1989.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 47B25, 47D45; Secondary 81C10, 81C12.

Key words and phrases. Self-adjointness, generation of unitary group, perturbation, momentum operator, Hamiltonian.

To begin with we study fundamental properties of p . A simple calculation shows that p is symmetric in $(L^2(\mathbf{R}^1), (\cdot, \cdot))$. It is easy to verify that $C_0^\infty(Q)$ with $Q = \mathbf{R}^1 - \{0\}$ is dense in $L^2(\mathbf{R}^1)$ and hence so is V .

Lemma 1. *The following hold for $A = ip$:*

- (i) *The domain $D(A)$ ($= V$) of A is dense in $L^2(\mathbf{R}^1)$;*
- (ii) *$\operatorname{Re}(Au, u) = 0$ for $u \in D(A)$.*

In the following, we shall show with the aid of the Hille-Yosida theorem that A generates a strongly continuous unitary group on $L^2(\mathbf{R}^1)$ to obtain the self-adjointness of $p (= -iA)$ by Stone's theorem. To examine the range $R(\lambda - A)$ of $\lambda - A$, we deal with the equation

$$(1) \quad \lambda u - Du + (c/x)Ru = f, \quad D = d/dx,$$

for given $\lambda \in Q$ and $f \in L^2(\mathbf{R}^1)$. If there exist an even function u_1 and an odd function u_2 satisfying

$$(2) \quad \begin{cases} \lambda u_1 - (D + c/x)u_2 = (f + Rf)/2, \\ \lambda u_2 - (D - c/x)u_1 = (f - Rf)/2, \end{cases}$$

then $u_1 + u_2$ is a solution of (1) since $u_1 - u_2 = R(u_1 + u_2)$. Moreover it is expected that for u_1 and u_2

$$\begin{aligned} \lambda^2 u_1 - (D + c/x)(D - c/x)u_1 &= \lambda \bar{f} + (D + c/x)\underline{f}, \\ \lambda^2 u_2 - (D - c/x)(D + c/x)u_2 &= \lambda \underline{f} + (D - c/x)\bar{f} \end{aligned}$$

hold true in some weak sense, where

$$(3) \quad \bar{f} = (f + Rf)/2 \quad \text{and} \quad \underline{f} = (f - Rf)/2.$$

Thus our main task is to look for such functions as u_1 and u_2 in the above.

2. SELF-ADJOINTNESS OF THE OPERATOR p

Let us consider the symmetric sesquilinear forms on $V \times V$:

$$\begin{aligned} b_1(u, v) &= \lambda^2(u, v) + (Du, Dv) + (c^2 - c)(u/x, v/x), \\ b_2(u, v) &= \lambda^2(u, v) + (Du, Dv) + (c^2 + c)(u/x, v/x). \end{aligned}$$

If $|c| > 1$, V becomes a Hilbert space with each of the inner products $b_1(\cdot, \cdot)$ and $b_2(\cdot, \cdot)$ for a fixed $\lambda \in Q$.

Let V_0 be the closure in V of $C_0^\infty(Q)$. Then we have

$$\begin{aligned} b_1(u, v) &= \lambda^2(u, v) + ((D - c/x)u, (D - c/x)v), \\ b_2(u, v) &= \lambda^2(u, v) + ((D + c/x)u, (D + c/x)v) \end{aligned}$$

for $u, v \in C_0^\infty(Q)$ and hence for $u, v \in V_0$. Using Riesz' representation theorem, we have that for given $\lambda \in Q$ and $f \in L^2(\mathbf{R}^1)$ with $\bar{f}$ and $\underline{f}$ defined

by (3), there exist unique u_1 and u_2 in V_0 satisfying, for all $\phi, \psi \in V_0$,

$$(4) \quad \begin{cases} b_1(u_1, \phi) = \lambda(\bar{f}, \phi) - (\underline{f}, (D - c/x)\phi), \\ b_2(u_2, \psi) = \lambda(\underline{f}, \psi) - (\bar{f}, (D + c/x)\psi). \end{cases}$$

Here we have used that $\lambda(\bar{f}, \cdot) - (\underline{f}, (D - c/x)\cdot)$ and $\lambda(\underline{f}, \cdot) - (\bar{f}, (D + c/x)\cdot)$ are continuous conjugate linear functionals on V_0 .

Let us here examine the properties of u_1 and u_2 to prove the following:

Lemma 2. *If $|c| > 1$, then we have*

$$R(\lambda - A) = L^2(\mathbf{R}^1) \quad \text{for } \lambda \in Q.$$

Proof. The proof is divided into three steps:

Step 1. First we shall show that u_1 is even and u_2 is odd. In view of $RD = -DR$ and $R(c/x) = -(c/x)R$, we see

$$\begin{aligned} b_1(Ru_1, \phi) &= b_1(u_1, R\phi) \\ &= \lambda(\bar{f}, \phi) - (\underline{f}, (D - c/x)\phi), \end{aligned}$$

$$\begin{aligned} b_2(-Ru_2, \psi) &= -b_2(u_2, R\psi) \\ &= \lambda(\underline{f}, \psi) - (\bar{f}, (D + c/x)\psi) \end{aligned}$$

for all $\phi, \psi \in V_0$. Thus the uniqueness of u_1 and u_2 gives

$$Ru_1 = u_1 \quad \text{and} \quad Ru_2 = -u_2.$$

Step 2. Next we will show that v_1 and v_2 defined by

$$(5) \quad \begin{cases} v_1 = \lambda^{-1}(\bar{f} + (D + c/x)u_2), \\ v_2 = \lambda^{-1}(\underline{f} + (D - c/x)u_1) \end{cases}$$

satisfy, for all $\phi \in C_0^\infty(Q)$,

$$(6) \quad \begin{cases} (u_1 - v_1, \lambda^2\phi + L(c^2 - c)\phi) = 0, \\ (u_2 - v_2, \lambda^2\phi + L(c^2 + c)\phi) = 0, \end{cases}$$

where $L(k) = -D^2 + k/x^2$ for a real constant k .

Indeed, substituting (5) into (4), we have

$$\begin{aligned} \lambda(u_1, \phi) + (v_2, (D - c/x)\phi) &= (\bar{f}, \phi), \\ \lambda(u_2, \psi) + (v_1, (D + c/x)\psi) &= (\underline{f}, \psi). \end{aligned}$$

On the other hand, we obtain directly from (5)

$$\begin{aligned} \lambda(v_1, \phi) + (u_2, (D - c/x)\phi) &= (\bar{f}, \phi), \\ \lambda(v_2, \psi) + (u_1, (D + c/x)\psi) &= (\underline{f}, \psi) \end{aligned}$$

by integration by parts. Thus we find

$$(7) \quad \begin{cases} \lambda(u_1 - v_1, \phi) = (u_2 - v_2, (D - c/x)\phi), \\ \lambda(u_2 - v_2, \psi) = (u_1 - v_1, (D + c/x)\psi) \end{cases}$$

for all $\phi, \psi \in V_0$, from which (6) follows.

Step 3. Finally we prove that $v_1 = u_1$ and $v_2 = u_2$, that is, u_1 and u_2 satisfy (2).

In the case of $c > 1$, since $c^2 + c > 2$, the following lemma is applicable to our operator $L(c^2 + c)$:

Lemma 3. *If $k \geq 3/4$, then $L(k) = -D^2 + k/x^2$ is essentially self-adjoint on $C_0^\infty(Q)$.*

For the proof, see e.g. Kato [2, pp. 402–403] or Okazawa [4, pp. 697–698]. From this lemma we see that the adjoint operator $L^*(k)$ of $L(k)$ for $k \geq 3/4$ coincides with the smallest closed extension of $L(k)$, and satisfies

$$(8) \quad R(\lambda^2 + L(k)^*) = L^2(\mathbf{R}^1) \quad \text{for } \lambda \in Q.$$

Let us continue the proof of Lemma 2, Step 3. Applying (8) with $k = c^2 + c$ to the second equality of (6), we obtain $v_2 = u_2$. Hence the first of (7) becomes $(u_1 - v_1, \phi) = 0$ for all $\phi \in V_0$. Since V_0 is dense in $L^2(\mathbf{R}^1)$, we obtain $v_1 = u_1$.

In the case of $c < -1$, since $c^2 - c > 2$, a similar argument by the first of (6), the second of (7), and (8) with $k = c^2 - c$ also gives $v_1 = u_1$ and then $v_2 = u_2$. Q.E.D.

Thus we have proved Theorem 1. In fact, as long as $|c| > 1$, $-iA (= p)$ is self-adjoint if and only if Lemmas 1 and 2 are true. See e.g. Goldstein [1, p. 32].

3. APPLICATION TO THE HAMILTONIAN H

Let us consider the operator H given by

$$H = (p^2 + x^2)/2.$$

We have shown in the above the self-adjointness of p and hence of p^2 with domain $D(p^2) = \{u \in V : pu \in V\}$ under the restriction $|c| > 1$. Clearly x^2 is self-adjoint on $D(x^2) = \{u \in L^2(\mathbf{R}^1) : x^2 u \in L^2(\mathbf{R}^1)\}$.

As mentioned in the introduction, we shall discuss the self-adjointness of H as an application of the result for p , and then as a consequence of a similar argument for iH . In this section Okazawa's theory [4] plays an important role.

The following is a corollary of Theorem 1:

Theorem 2. *If $|c| > 1$, then H is self-adjoint on*

$$W_1 = \{u \in D(p^2) : x^2 u \in L^2(\mathbf{R}^1)\}.$$

Proof. We will apply Okazawa's perturbation theorem [4, Theorem 5.4] to the nonnegative self-adjoint operators p^2 and x^2 . An elementary calculation gives

$$(9) \quad \operatorname{Re}(p^2 u, n x^2 (n + x^2)^{-1} u) \geq -(u, u)$$

for $u \in D(p^2)$ and $n = 1, 2, \dots$, where $n x^2 (n + x^2)^{-1}$ is the Yosida-approximation of x^2 . Indeed, the left-hand side equals

$$(pu, n x^2 (n + x^2)^{-1} pu) + \operatorname{Im}(pu, 2n^2 x (n + x^2)^{-2} u).$$

Thus H is self-adjoint on $W_1 (= D(p^2) \cap D(x^2))$. Q.E.D.

It is not very easy to deal with the domain W_1 , which is indeed a subset of the abstract set $D(p^2)$. Let us observe here that

$$p^2 u = -D^2 u + (c/x^2)(c - R)u$$

holds for all $u \in C_0^\infty(Q)$. Therefore we may use the same notation H to denote the operator

$$H = \{-D^2 + (c/x^2)(c - R) + x^2\}/2.$$

Recalling the technique used for ip in the preceding sections, we can establish the self-adjointness of this operator H on the simpler domain

$$W_2 = \{u \in H^2(\mathbf{R}^1) : u/x^2, x^2 u \in L^2(\mathbf{R}^1)\},$$

however, under somewhat stronger condition on c :

Theorem 3. *If $|c| > 3/2$, then H is self-adjoint on W_2 .*

Proof. An easy calculation shows that Lemma 1 holds true with $A = iH$. To prove Lemma 2 with $A = iH$, we will look for w_1 and w_2 in W_2 satisfying (cf. (2))

$$(10) \quad \begin{cases} \lambda w_1 - i\{L(c^2 - c) + x^2\}w_1 = \bar{f}, \\ \lambda w_2 - i\{L(c^2 + c) + x^2\}w_2 = \underline{f}, \end{cases}$$

respectively, for given $\lambda \in Q$ and $f \in L^2(\mathbf{R}^1)$.

Since $|c| > 3/2$, both $L(c^2 + c)$ and $L(c^2 - c)$ are self-adjoint on the set

$$D(L) = \{u \in H^2(\mathbf{R}^1) : u/x^2 \in L^2(\mathbf{R}^1)\}$$

by the following lemma due to Okazawa [4, Theorem 6.8]:

Lemma 4. *If $k > 3/4$, then $L(k) = -D^2 + k/x^2$ is self-adjoint on $D(L)$.*

Let us turn to the proof of Theorem 3. Noting

$$\begin{aligned} & (L(k)u, nx^2(n + x^2)^{-1}u) \\ &= (Du, nx^2(n + x^2)^{-1}Du) + (Du, 2n^2x(n + x^2)^{-2}u) + (ku, n(n + x^2)^{-1}u), \end{aligned}$$

we obtain the estimate similar to (9):

$$\operatorname{Re}(L(k)u, nx^2(n + x^2)^{-1}u) \geq -(u, u)$$

for $u \in D(L)$ and $n = 1, 2, \dots$. Hence Okazawa's perturbation theorem again gives the self-adjointness on $W_2 (= D(L) \cap D(x^2))$ of $L(k) + x^2$ with $k > 3/4$. The operators $L(c^2 + c) + x^2$ and $L(c^2 - c) + x^2$ are now self-adjoint and, by Stone's theorem, there exist unique w_1 and w_2 in W_2 satisfying (10). Evidently w_1 is even and w_2 is odd, and $w_1 + w_2$ is a solution of

$$\lambda w - 2iHw = f$$

for every $\lambda \in Q$ and $f \in L^2(\mathbf{R}^1)$. Therefore, as long as $|c| > 3/2$, $R(\lambda - iH)$ equals $L^2(\mathbf{R}^1)$ for $\lambda \in Q$.

Thus we have proved Lemmas 1 and 2 with $A = iH$ and $|c| > 3/2$, and equivalently Theorem 3. Q.E.D.

As for the Hamiltonian H together with the momentum operator p , we refer the reader to the recent paper [8].

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