AN EXAMPLE ON NULL SETS OF PARABOLIC MEASURES

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ABSTRACT. A set E on the real line of Hausdorff dimension 1 is constructed, such that the graph of E on the boundary of any C^2 domain $\{t > \tau(x)\}$, or on the boundary of any $\text{Lip } \frac{1}{2}$ domain $\{x > \chi(t)\}$, is null with respect to the parabolic measure associated with any parabolic operator $L = a(x, t)\partial^2/\partial x^2 - \partial/\partial t$ on $\mathbf{R}^2$; a is Hölder continuous and $0 < \Lambda_1 \leq a \leq \Lambda_2 < \infty$ for some constants Λ_1 and Λ_2 .

This note concerns parabolic operators L on $\mathbf{R}^2$ of the form

$$(0.1) \quad L = a(x, t) \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t},$$

where a is Hölder continuous and satisfies

$$(0.2) \quad 0 < \Lambda_1 \leq a(x, t) \leq \Lambda_2 < \infty$$

for some constants Λ_1 and Λ_2 .

For such operators, the maximum principle on bounded domains and the Harnack principle are classical. The Dirichlet problem is solvable on rectangular domains—the base is regular because $a(x, t) \leq \Lambda_2$ and the sides are regular because $a(x, t) \geq \Lambda_1$.

In a domain Ω defined by $\{t > \tau(x)\}$ with τ continuous on $-\infty < x < \infty$, or defined by $\{x \leq \chi(t)\}$ with $\chi \in \text{Lip } \frac{1}{2}$ on $-\infty < t < \infty$, the maximum principle is still valid; and the Dirichlet problem, with continuous, compactly supported boundary data, is uniquely solvable (to be verified in §6). Therefore, the value of a solution at a point (x, t) can be treated as a linear functional on the space of the boundary values, and can be represented by a probability measure $\omega_{\Omega}^{(x, t)}$ on the boundary. It is called the parabolic measure associated with L ; and for a Borel set $E \subseteq \partial\Omega$, $\omega_{\Omega}^{(x, t)}(E)$ is a solution of $L = 0$ on Ω .

In [1], Fabes and Kenig proved that on the domain $\{t > 0\}$, there exists a parabolic operator L , with coefficient a uniformly continuous in $\{t > 0\}$, so that the associated parabolic measure is purely singular with respect to the Lebesgue measure on $\{t = 0\}$.

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It is proved in [4] that on the domain $\{t > 0\}$, for any $\delta > 0$, there exists a parabolic operator L , such that the corresponding parabolic measure is concentrated on a set of dimension $< \delta$. It is not clear whether a similar result holds on $\{x > 0\}$. However, there exist a $\text{Lip } \frac{1}{2}$ domain $\{x > \chi(t)\}$ and a set E on the boundary, of dimension < 1 , so that all the parabolic measures associated with the heat equation vanish off E ([3]).

On half planes $\{t > -Mx\}$ or $\{x > 0\}$, all boundary sets of zero Λ_1/Λ_2 -dimensional Hausdorff measure have zero parabolic measure associated with any parabolic operator satisfying (0.2)—a consequence of Lemma D below and the maximum principle.

In this note, we construct a set E on $\mathbf{R}$ of Hausdorff dimension 1, such that the graph of E on the boundary of any C^2 domain $\{t > \tau(x)\}$ (or of any $\text{Lip } \frac{1}{2}$ domain $\{x > \chi(t)\}$) is null with respect to the parabolic measure associated with any parabolic operator L . More precisely, we prove

Theorem 1. *Given any Hausdorff measure function $h(s)$ with $h(s)/s \rightarrow +\infty$ as $s \rightarrow 0^+$, there exists a compact set E on $\mathbf{R}$ of infinite Hausdorff measure $\tau(x) C^2 \text{Lambda}_h(E)$, such that on any domain $\{t > \tau(x)\}$, with $t = \tau(x) C^2$ on $-\infty < x < \infty$, (respectively, $\{x > \chi(t)\}$, with $x = \chi(t) \text{Lip } \frac{1}{2}$ on $-\infty < t < \infty$), the parabolic measure associated with any parabolic operator L vanishes on $\{(x, \tau(x)) : x \in E\}$ (respectively, $\{(\chi(t), t) : t \in E\}$).*

On C^2 domains $\{t > \tau(x)\}$, or domains $\{x > \chi(t)\}$ slightly better than $\text{Lip } \frac{1}{2}$, the parabolic measure associated with the heat equation is known to be mutually absolutely continuous with respect to the Lebesgue measure of the projection of the boundary set onto $t = 0$, or $x = 0$ respectively ([5], [6]).

In §2–5, we give the proof of Theorem 1; in §6, we substantiate the existence and the uniqueness of the Dirichlet problem, which have some independent interest. Propositions 2 and 3 are about parabolic measures on C^2 and $\text{Lip } \frac{1}{2}$ domains, and are workable substitutes for the doubling property.

The author is grateful to the referee for the following remark: when the domain is $\{t > \tau(x)\}$ or $\{x > 0\}$, the theorem is valid if $a(x, t)$ is only uniformly continuous—parabolic measures exist under these circumstances. In fact, smooth compactly supported functions on the boundary have extensions to $\mathbf{R}^2$ which lie in the Sobolev space of functions with one time derivative and two space derivatives in L^2 . Based on a lemma of Stroock and Varadhan [1, Lemma 0], the solution of the Dirichlet problem exists in this Sobolev space. The maximum principle follows by smoothing the coefficients and taking the limit of the solutions.

1. THE SET E

Let h be a Hausdorff function with $h(s)/s \rightarrow \infty$ as $s \rightarrow 0^+$. Replacing $h(s)$ by $\inf_{0 < t \leq 1} h(ts)/t$ if necessary, we may assume that $h(s)/s$ is decreasing. We choose a continuous increasing function f on $[0, 1]$ with $f(0) = 0, f(\frac{1}{2}) = 1$

and $h(sf(s))/s \rightarrow \infty$ as $s \rightarrow 0^+$; and integers $j(n), n = 1, 2, 3, \dots$, so that $j(1) = 1$ and $f(2^{-j(n)})/f(2^{-j(n-1)}) \leq 1/n$. We define $\alpha_j = \frac{1}{2}$ for $j \neq j(n), n = 1, 2, 3, \dots$, and choose numbers $\alpha_{j(n)}$ inductively so that

$$\prod_1^{j(n)} \alpha_j = 2^{-j(n)} f(2^{-j(n)}), n = 1, 2, \dots.$$

Note that $\alpha_1 = \frac{1}{2}$,

$$\alpha_{j(n)} \leq \frac{1}{2n}, \quad n = 1, 2, \dots,$$

and

$$(1.1) \quad h \left(\prod_1^{j(n)} \alpha_j \right) 2^{j(n)} \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

Let $\mathcal{F}_0 = [0, 1]$. And let $\mathcal{F}_k$ ($k \geq 1$) consist of 2^k closed intervals in $[0, 1]$ with mutually disjoint interiors, of length $\prod_1^k \alpha_j$ each, constructed inductively as follows: for each interval I in $\mathcal{F}_{k-1}$, we subdivide I into two equal intervals if $\alpha_k = \frac{1}{2}$; subdivide the middle $2\alpha_k$ portion of I into two equal intervals if $\alpha_k < \frac{1}{2}$. These new intervals constitute $\mathcal{F}_k$. Let

$$E = \bigcap_1^\infty \mathcal{F}_k.$$

It is clear from (1.1) that the Hausdorff measure $\Lambda_h(E) = \infty$.

Denote by $\Delta(x, r) = \{y: |y - x| < r\}$ and we have

Proposition 1. *Let E be the set constructed above and μ be a measure on $[0, 1]$. Suppose that there exist $K > 1, \varepsilon_0 > 0$, a closed set Z on $[0, 1]$, and a function α which is positive continuous on $[0, 1] \setminus Z$, and equals one on Z , such that for any $0 < \varepsilon < \varepsilon_0, 0 \leq x \leq 1$, and $0 < r\varepsilon^{-K} < R \leq \alpha(x)$,*

$$(1.2) \quad \mu(\Delta(x, r)) \leq C\varepsilon\mu(\Delta(x, R)),$$

where C is a constant independent of x, ε, r, R , and α . Then $\mu(E) = 0$.

Proof. Given $0 < \varepsilon < 1$, let $N = \lceil \varepsilon^{-K-10} \rceil$, where $\lceil \cdot \rceil$ is the greatest integer function. We cover E by a collection $\mathcal{J}$ of intervals from $\bigcup_{n \geq N} \mathcal{F}_{j(n)-1}$ with mutually disjoint interiors, so that if $I \in \mathcal{J} \cap \mathcal{F}_{j(n)-1}$ and J_I^1 and J_I^2 are the two intervals in $\mathcal{F}_{j(n)}$ that are contained in I , then either (a) $(J_I^1 \cup J_I^2) \cap Z \neq \emptyset$, or (b) $(J_I^1 \cup J_I^2) \cap Z = \emptyset$ and $|I| \leq \inf_I \alpha(x)$. This is possible because $\alpha(x)$ is positive continuous on $[0, 1] \setminus Z$.

Let $\mathcal{J}^k = \{J_I^k: I \in \mathcal{J}, k = 1, 2\}$. Clearly, $\mathcal{J}$ is a covering of E and $|J_I^k|/|I| \leq 1/N < \varepsilon^{K+9}$ for any $I \in \mathcal{J}$. Therefore from (1.2) we obtain

$$\mu(J_I^k) \leq C\varepsilon\mu(I) \quad \text{for } I \in \mathcal{J}, k = 1, 2.$$

Thus $\mu(E) \leq C\varepsilon$. Hence $\mu(E) = 0$.

2. PROOF OF THEOREM 1

It is apparent that we only need to prove that the parabolic measures in question have the properties of μ in Proposition 1. Fix a parabolic operator L on $\mathbf{R}^2$ satisfying (0.2).

Let $t = \tau(x)$ be a C^2 function on $\mathbf{R}$, $\Omega = \{t > \tau(x)\}$ and ω be the parabolic measure associated with L on $\partial\Omega$. Because parabolic measure zero is a local property, we assume, as we may, that $\sup |\tau''| < +\infty$; let $A = 1 + \sup |\tau''|$ and

$$\alpha(x) = \begin{cases} 1, & \tau'(x) = 0, \\ \min\{1, \frac{|\tau'(x)|}{400A^2(\Lambda_1+1)}\}, & \tau'(x) \neq 0. \end{cases}$$

We denote by $\Phi(x, r) = \{(y, \tau(y)) : |y - x| < r\}$ and by capital C positive constants depending at most on Λ_1, Λ_2 , and A . We have

Proposition 2. *Under the above assumptions, let $0 < \varepsilon < 10^{-5}$, $x_0 \in \mathbf{R}$, $0 < r < R < \alpha(x_0)$, and*

$$(2.1) \quad \frac{r}{R} < \frac{\Lambda_1}{\Lambda_2} \varepsilon^{2+\Lambda_2/\Lambda_1}.$$

Then

$$(2.2) \quad \omega^{(x,t)}(\Phi(x_0, r)) \leq C\varepsilon \omega^{(x,t)}(\Phi(x_0, R))$$

for $(x, t) \in \Omega \setminus \Gamma$, where

$$\begin{aligned} \Gamma &= \{(x, t) : |x - x_0| < \varepsilon R, |t - \tau(x_0)| < (A + \frac{1}{\Lambda_1})R^2\}, \quad \tau'(x_0) = 0; \\ &= \left\{ (x, t) : |x - x_0| < \sqrt{\Lambda_1 |\tau'(x_0)|} R/4, \right. \\ &\quad \left. |t - \tau(x_0)| < \left| \tau \left(x_0 - \frac{\Lambda_1 R}{10^4 \Lambda_2 \log \frac{1}{\varepsilon}} \right) - \tau(x_0) \right| \right\}, \quad \tau'(x_0) \neq 0. \end{aligned}$$

In particular, for any x_0 in a closed interval $[x_1, x_2]$, (2.2) holds for every $(x, t) \in \Omega$ with

$$t > \tau(x_1) + 2(1/\Lambda_1 + |\tau'(x_1)| + A)(1 + |x_2 - x_1|^2).$$

Fix a $\text{Lip } \frac{1}{2}$ function $x = \chi(t)$ on $\mathbf{R}$ with

$$(2.3) \quad |\chi(t) - \chi(s)| \leq B|t - s|^{1/2}, \quad t, s \in \mathbf{R},$$

for some $B > 0$, denote by $\tilde{\Omega}$ the domain $\{x > \chi(t)\}$, and let $\tilde{\omega}$ be the parabolic measure associated with L on $\partial\tilde{\Omega}$. Denote by $\Psi(t, r) = \{(\chi(s), s) : |s - t| < r\}$ and by lowercase c positive constants depending at most on Λ_1, Λ_2 , and B . We have

Proposition 3. *Under the above assumptions on the $\text{Lip } \frac{1}{2}$ domain $\{x > \chi(t)\}$, let $0 < \varepsilon < B^2/4000(\Lambda_2 + B^2)$, $x_0 \in \mathbf{R}$, $0 < r < R < 1$ and*

$$(2.4) \quad r/R < e^{2+2\Lambda_2/\Lambda_1}.$$

Then

$$(2.5) \quad \tilde{\omega}^{(x,t)}(\Psi(t_0, r)) \leq c\varepsilon \tilde{\omega}^{(x,t)}(\Psi(t_0, R))$$

for $(x, t) \in \tilde{\Omega} \setminus \tilde{\Gamma}$, where

$$\tilde{\Gamma} = \left\{ (x, t): |x - \chi(t_0)| < B\sqrt{R}/2 \text{ and } -r < t - t_0 < \frac{B^2 R}{4000(\Lambda_2 \log \frac{1}{\varepsilon} + B^2)} \right\}.$$

In particular, for t_0 in a closed interval $[t_1, t_2]$, (2.4) holds for all $(x, t) \in \tilde{\Omega}$ with $t > 1 + t_2$.

It follows immediately from Propositions 1, 2, and 3 and the Harnack principle that $\omega(\{(x, \tau(x)): x \in E\}) \equiv 0$ on Ω , and that $\tilde{\omega}(\{(x(t), t): t \in E\}) \equiv 0$ on $\tilde{\Omega}$. Thus Theorem 1 is proved.

In the C^2 case, the comparison (2.2) is not uniform with regard to the size of R . The restriction $R < \alpha(x_0)$ is very inconvenient; and we believe that it could be removed.

We prove Propositions 2 and 3 in §§3–5 below.

3. PRELIMINARY LEMMAS

We let L be a parabolic operator on $\mathbf{R}^2$; and in any domain U , denote by ω_U the associated parabolic measure. The next three lemmas are proved in [4]; the restriction that $a(x, t)$ is C^2 there can easily be removed.

Lemma A. *Let $U = \{x > 0, t > 0\}$, $b > 0$ and $0 < t < b^2/12\Lambda_2$. Then $\omega_U^{(b,t)}(\{x = 0\}) < e^{-b^2/144t\Lambda_2}$.*

Lemma B. *Let $U = \{x > 0, t > 0\}$ and $0 < \varepsilon < 1$. Then $\omega_U^{(x,t)}(\{x = 0\}) > 1 - \varepsilon$, for $x > 0$ and $t > x^2(3\varepsilon^2\Lambda_1)^{-1}$.*

Lemma C. *Let $b_1 > b_2 > 0$, $U = \{-b_2 < x < b_1, t > 0\}$ and $S = \partial U \cap \{x = b_1, \cdot\}$; then $\omega_U^{(0,t)}(S) \leq b_2 b_1^{-1}$, for $t > 0$.*

Lemma D. *Corresponding to the parabolic operator L on $\mathbf{R}^2$, define*

$$(3.1) \quad F(x, t) = \begin{cases} t^{-\Lambda_2/2\Lambda_1} e^{-x^2/4\Lambda_1 t}, & \text{for } t > 0, \\ 0, & \text{for } t \leq 0; \end{cases}$$

$$(3.2) \quad G(x, t) = \begin{cases} t^{-\Lambda_1/2\Lambda_2} e^{-x^2/4\Lambda_2 t}, & \text{for } t > 0, \\ 0, & \text{for } t \leq 0; \end{cases}$$

$$(3.3) \quad H(x, t) = \begin{cases} \left(\frac{x}{t^{3/2}}\right)^{\Lambda_1/\Lambda_2} e^{-x^2/4\Lambda_2 t}, & \text{for } x > 0 \text{ and } t > 0, \\ 0, & \text{for } x \leq 0 \text{ or } t \leq 0; \end{cases}$$

and for $-\infty < M < \infty$, define

$$(3.4) \quad K_M(x, t) = \begin{cases} \left(\frac{t+Mx}{t^{3/2}}\right)^{\Lambda_1/\Lambda_2} e^{-x^2/4\Lambda_2 t}, & \text{for } t > 0 \text{ and } t > -Mx, \\ 0, & \text{for } t \leq 0 \text{ or } t \leq -Mx. \end{cases}$$

Then $LF \geq 0$ on $\mathbf{R}^2 \setminus \{(0, 0)\}$, $LG \leq 0$ on $\mathbf{R}^2 \setminus \{(0, 0)\}$, $LH \leq 0$ on $\{x > 0\}$, and $LK_M \leq 0$ on $\{t > -Mx\}$.

Proofs contain straightforward differentiations and tedious estimations using the quadratic formula.

4. C^2 DOMAINS $\{t > \tau(x)\}$

Let us retain all the assumptions from the second paragraph of §2, and assume, for convenience, that $\tau(0) = 0$.

To prove Proposition 2, we need the following lemmas.

Lemma 1. Suppose that $\tau'(0) = 0$. Let $0 < R < 1$ and $(x, t) \in \Omega \cap \{|x| \leq R/2$ and $t \leq (A + 1/\Lambda_1)R^2\}$. Then

$$(4.1) \quad \omega^{(x,t)}(\Phi(0, R)) > C.$$

Proof. Let $D = \{(y, s) : |y| < R, s > -AR^2/2\}$ and $S = \partial D \cap \{s = -AR^2/2\}$. It follows from Lemma A and the maximum principle that $\omega_D^{(y,s)}(S) > \frac{1}{2}$, when $|y| < R/2$ and $s < -AR^2/2 + R^2/1000\Lambda_2$. Hence $\omega_D^{(x,t)}(S) > C > 0$ by the Harnack principle. Since $\tau(x) > -AR^2/2$ for $|x| < R$, (4.1) follows from the maximum principle.

Lemma 2. Suppose that $\tau'(0) = 0$. Given $0 < \varepsilon < 1$, $0 < \rho < 1$, $|x| < \rho$ and $t > \rho^2(A/2 + 1/\varepsilon^2\Lambda_1)$, then $\omega_{\Omega \cap \{|x| < \rho\}}^{(x,t)}(\Phi(0, \rho)) < \varepsilon$.

Proof. Let $D = \{|x| < \rho$ and $t > \rho^2 A/2\}$. We deduce from the maximum principle and Lemma B that

$$\omega_{\Omega \cap \{|x| < \rho\}}^{(x,t)}(\Phi(0, \rho)) < \omega_D^{(x,t)}(\{|x| < \rho, t = \rho^2 A/2\}) < \varepsilon.$$

Lemma 3. Suppose that $\tau'(0) = 0$. Given $0 < \varepsilon, \rho < 1$ and $0 < r < \rho \varepsilon^{\Lambda_2/\Lambda_1}$, then $\omega(\Phi(0, r)) < C\varepsilon$ at points (ρ, t) and $(-\rho, t) \in \Omega$.

Proof. Let $u(x, t) = G(x, t + Ar^2)$, where G is the function defined in (3.2). We note that on $\Phi(0, r)$, $u \geq Cr^{-\Lambda_1/\Lambda_2}$; and that $u(\rho, t)$ attains its maximum value $(e\rho^2/2\Lambda_1)^{-\Lambda_1/2\Lambda_2}$ at $t = -Ar^2 + \rho^2/2\Lambda_1$. Because $u \geq 0$ and $Lu \leq 0$ in Ω , applying the maximum principle on Ω , we obtain

$$\omega^{(\rho,t)}(\Phi(0, r)) \leq Cr^{\Lambda_1/\Lambda_2} u(\rho, t) \leq C\left(\frac{r}{\rho}\right)^{\Lambda_1/\Lambda_2} < C\varepsilon.$$

Similarly, $\omega^{(-\rho, t)}(\Phi(0, r)) < C\varepsilon$.

Next, we assume that $\tau'(0)$ is negative, let

$$(4.2) \quad m \equiv -\tau'(0) > 0 \quad \text{and} \quad 0 < r, \rho, R < \min\{1, m/400A^2(\Lambda_1 + 1)\}.$$

It is clear that, for $|x| < m/2A$,

$$(4.3) \quad \tau'(x) \leq -\frac{m}{2} \quad \text{and} \quad |\tau(x) + mx| < m|x|/4.$$

Lemma 4. Assume (4.2), then $\omega^{(x, 0)}(\Phi(0, R)) > C$, where

$$0 < x < \sqrt{\Lambda_1 m R}/2.$$

Proof. Let $D = \{(x, t) : 0 < x < 10\sqrt{\Lambda_1 m R}, t > -3mR/8\}$, and $S = \{(0, t) : -3mR/8 < t < 0\}$. We note from (4.2) that $R < m/2A$ and $10\sqrt{\Lambda_1 m R} < m/2A$; thus it follows from (4.3) that $\tau(R/2) < -3mR/8$ and that $\tau'(x) < 0$ for $0 < x < 10\sqrt{\Lambda_1 m R}$. From the maximum principle, we see that for $0 < x < \sqrt{\Lambda_1 m R}/2$,

$$\begin{aligned} \omega^{(x, 0)}(\Phi(0, \frac{R}{2})) &> \omega_D^{(x, 0)}(S) \\ &> \omega_{\{x>0, t>-3mR/8\}}^{(x, 0)}(S) - \omega_D^{(x, 0)}(\{x = 10\sqrt{\Lambda_1 m R}\}) \\ &> \frac{1}{3} - \frac{1}{19}. \end{aligned}$$

The estimates $1/3$ and $1/19$ follow from Lemmas B and C respectively.

Lemma 5. Assume (4.2) and let $0 < \varepsilon < \frac{1}{4}$. Then $\omega^{(x, t)}(\Phi(0, \rho)) < \varepsilon$, when $(x, t) \in \Omega$ with $x \geq 24\sqrt{\Lambda_2 \log \frac{1}{\varepsilon} m \rho}$ and $\tau(\rho) < t < \tau(-\rho)$.

Proof. Let $D = \{(x, t) : x > \rho \text{ and } t > \tau(\rho)\}$ and $S = \{(\rho, t) : \tau(\rho) < t < \tau(-\rho)\}$. It follows from the maximum principle that $\omega^{(x, t)}(\Phi(0, \rho)) < \omega_D^{(x, t)}(S)$ for those (x, t) satisfying the conditions above. Since $\tau(-\rho) - \tau(\rho) < (5/2)m\rho$ and $\rho < m$, we conclude the lemma by applying Lemma A to $\omega_D(S)$.

Lemma 6. Assume (4.2) and let $0 < \varepsilon < 10^{-5}$. Suppose that $r < 10^4 \rho \varepsilon^{\Lambda_2/\Lambda_1}$, then $\omega^{(x, \tau(-\rho))}(\Phi(0, r)) < C\varepsilon$ for any $(x, \tau(-\rho)) \in \Omega \cap \{x > -\rho\}$.

Proof. Let $M = m/2$ in (3.4) and

$$u(x, t) = K_{m/2}(x + \sqrt{mr}, t + 2mr).$$

Then $u \geq 0$ and $Lu \leq 0$ in $\{t > -2mr \text{ and } t + 2mr > -(m/2)(x + \sqrt{mr})\}$; in particular, in $\Omega \cap \{\tau(r) < t < \tau(-\rho) \text{ and } x > -\rho\}$ because $\tau(x) > \min\{(-5/4)mx, (-3/4)mx\} > -mx/2 - m^{3/2}r/2 - 2mr$ on $-\rho < x < r$. Because $r < m/400$, it is easy to see that $u \geq Cr^{-\Lambda_1/\Lambda_2}$ on $\Phi(0, r)$. Applying the maximum principle on $\Omega \cap \{x > -\rho \text{ and } \tau(r) < t < \tau(-\rho)\}$, we obtain

$$\omega^{(x, \tau(-\rho))}(\Phi(0, r)) \leq Cr^{\Lambda_1/\Lambda_2} u(x, \tau(-\rho)).$$

Let $\delta = \tau(-\rho) + 2mr$, $X = x + \sqrt{mr}$ and note that $m\rho/4 < \delta < 4m\rho$. Then we have, for $x > -\rho$,

$$\begin{aligned} u(x, \tau(-\rho)) &= \left(\frac{\delta + \frac{m}{2}X}{\delta^{3/2}} \right)^{\Lambda_1/\Lambda_2} e^{-X^2/4\Lambda_2\delta} \\ &\leq C \left[\left(\frac{1}{\delta} \right)^{\Lambda_1/2\Lambda_2} + \left(\frac{m}{\delta} \right)^{\Lambda_1/\Lambda_2} \left(\frac{|X|}{\delta^{1/2}} \right)^{\Lambda_1/\Lambda_2} e^{-X^2/4\Lambda_2\delta} \right] \\ &\leq C \left[\left(\frac{1}{\delta} \right)^{\Lambda_1/2\Lambda_2} + \left(\frac{m}{\delta} \right)^{\Lambda_1/\Lambda_2} \right] \\ &\leq C \left[\left(\frac{1}{m\rho} \right)^{\Lambda_1/2\Lambda_2} + \left(\frac{1}{\rho} \right)^{\Lambda_1/\Lambda_2} \right] \leq C \left(\frac{1}{\rho} \right)^{\Lambda_1/\Lambda_2}. \end{aligned}$$

If $r/\rho < 10^4 \varepsilon^{\Lambda_2/\Lambda_1}$, then

$$\omega^{(x, \tau(-\rho))}(\Phi(0, r)) \leq C \left(\frac{r}{\rho} \right)^{\Lambda_1/\Lambda_2} < c\varepsilon.$$

Proof of Proposition 2. Because (2.1) is a local property, we assume, as we may, that $x_0 = 0$ and recall that $\tau(0) = 0$. Let ε, r , and R satisfy the conditions in Proposition 2.

First, we assume that $\tau'(0) = 0$. Let $\ell_1 = \{x = \varepsilon R, \tau(\varepsilon R) < t < (A + 1/\Lambda_1)R^2\}$, $\ell_2 = \{x = -\varepsilon R, \tau(-\varepsilon R) < t < (A + 1/\Lambda_1)R^2\}$ and $\ell_3 = \{-\varepsilon R \leq x \leq \varepsilon R, t = (A + 1/\Lambda_1)R^2\}$. It is easy to verify that $\Omega \cap \partial\Gamma = \ell_1 \cup \ell_2 \cup \ell_3$. Because of the maximum principle, it suffices to verify (2.2) on $\ell_1 \cup \ell_2 \cup \ell_3$. We deduce from Lemma 1 that

$$\omega^{(x, t)}(\Phi(0, R)) > C \quad \text{on } \ell_1 \cup \ell_2 \cup \ell_3.$$

Letting $\rho = \varepsilon R$, we deduce from (2.1) and Lemma 3 that

$$\omega^{(x, t)}(\Phi(0, r)) < \varepsilon \quad \text{on } \ell_1 \cup \ell_2.$$

Finally, for $(x, t) \in \ell_3$, it follows from the Markov property, the maximum principle and Lemmas 2 and 3 that

$$\begin{aligned} \omega^{(x, t)}(\Phi(0, r)) &= \omega_{\Gamma}^{(x, t)}(\Phi(0, r)) + \int_{\ell_1 \cup \ell_2} \omega^{(y, s)}(\Phi(0, r)) d\omega_{\Gamma}^{(x, t)}(y, s) \\ &\leq \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

Thus (2.2) holds, when $\tau'(0) = 0$.

Next, we assume that $\tau'(0) = -m < 0$. (The remaining case $\tau'(0) > 0$ can be treated similarly.) Let $\rho = \Lambda_1 R / 10^4 \Lambda_2 \log 1/\varepsilon$, $s_1 = \{x = \sqrt{\Lambda_1 m R / 4}, \tau(r) < t \leq \tau(-\rho)\}$ and $s_2 = \{t = \tau(-\rho), -\rho < x \leq \sqrt{\Lambda_1 m R / 4}\}$. It is clear that $\partial\Gamma \cap \Omega \cap \{t > \tau(r)\} = s_1 \cup s_2$. By the maximum principle and the fact that $\omega^{(x, t)}(\Phi(0, r)) = 0$ for $t \leq \tau(r)$, we need only to prove (2.2) on $s_1 \cup s_2$. The

x -distance from any point on $s_1 \cup s_2$ to $\partial\Omega$ is at most $\rho + \sqrt{\Lambda_1 mR}/4 < \sqrt{\Lambda_1 mR}/2$. We note from Lemma 4 that

$$\omega^{(x,t)}(\Phi(0, R)) > C \quad \text{on } s_1 \cup s_2.$$

Because $r < (\Lambda_1/\Lambda_2)R\varepsilon^{2+\Lambda_2/\Lambda_1} < 10^4 \rho \varepsilon^{\Lambda_2/\Lambda_1}$, we obtain

$$(4.4) \quad \omega^{(x, \tau(-\rho))}(\Phi(0, r)) < C\varepsilon$$

on s_2 , from Lemma 6. Since $\sqrt{\Lambda_1 mR}/4 > 24\sqrt{\Lambda_2 \log \frac{1}{\varepsilon} m\rho}$, (4.4) holds also on s_1 . The conclusion (2.2) thus follows. And this completes the proof of Proposition 2.

The proof above is very lengthy, because we are unable to combine the case $\tau'(0) \neq 0$, with the case $\tau'(0) = 0$. When $\tau'(0) \neq 0$, implicitly we regard x as a function of t on $\partial\Omega$ near $(0,0)$.

5. Lip $\frac{1}{2}$ DOMAINS $\{x > \chi(t)\}$

We retain the assumptions from the third paragraph of §2, and assume that $\chi(0) = 0$. The structure of the proof of Proposition 3 is the same as that of Proposition 2.

Lemma 7. Assume that $0 < R < 1$ and that $0 < x < 2B\sqrt{R}$. Then

$$(5.1) \quad \tilde{\omega}^{(x,0)}(\Psi(0, R)) > c.$$

Proof. Let $D = \{x > -B\sqrt{R}, t > -R\}$ and $S = \{x = -B\sqrt{R}, -R < t < 0\}$. From (2.3), the maximum principle and the Harnack principle, it follows that

$$\tilde{\omega}^{(x,0)}(\Phi(0, R)) > \omega_D^{(x,0)}(S) \geq c\omega_D^Q(S),$$

where $Q = (-B\sqrt{R} + \sqrt{\Lambda_1 R}/2, -R/2)$. From Lemma B, we deduce that $\omega_D^Q(S) > \frac{1}{2}$. Therefore (5.1) holds.

Lemma 8. Given $0 < \varepsilon < \frac{1}{4}$, $0 < \delta < 1$ and $-\delta < t < \delta$, then

$$\tilde{\omega}^{(x,t)}(\Psi(0, \delta)) < \varepsilon$$

if $x > (20\sqrt{\Lambda_2 \log 1/\varepsilon} + B)\sqrt{\delta}$.

Proof. Let $D = \{x > B\sqrt{\delta}, t > -\delta\}$ and $S = \{x = B\sqrt{\delta}, -\delta < t < \delta\}$. It follows from (2.3), Lemma A and the maximum principle that $\tilde{\omega}^{(x,t)}(\Psi(0, \delta)) < \omega_D^{(x,t)}(S) < \varepsilon$.

Lemma 9. Suppose that $0 < \varepsilon < 1$, $0 < \delta < 1$ and that $r < \delta \varepsilon^{2\Lambda_2/\Lambda_1}$; then $\tilde{\omega}^{(x,\delta)}(\Psi(0, r)) < c\varepsilon$ for any $(x, \delta) \in \tilde{\Omega}$.

Proof. Let $u(x, t) = G(x, t + 2r)$, with G defined in (3.2). Because of (2.3), we have $u(x, t) \geq cr^{-\Lambda_1/2\Lambda_2}$ on $\Psi(0, r)$. Applying the maximum principle on

$\tilde{\Omega} \cap \{t > -r\}$, we obtain that for $(x, \delta) \in \tilde{\Omega}$,

$$\tilde{\omega}^{(x, \delta)}(\Psi(0, r)) \leq cr^{\Lambda_1/2\Lambda_2} u(x, \delta) \leq cr^{\Lambda_1/2\Lambda_2} (\delta + 2r)^{-\Lambda_1/2\Lambda_2} < c\varepsilon.$$

Proof of Proposition 3. We assume, as we may, that $t_0 = 0$ and $\chi(0) = 0$. Let ε, r , and R be as given and let $\delta = B^2 R / 4000(\Lambda_2 \log 1/\varepsilon + B^2)$, $\ell_1 = \{t = \delta, \chi(\delta) < x < B\sqrt{R}/2\}$ and $\ell_2 = \{x = B\sqrt{R}/2, -r < t \leq \delta\}$. Clearly $\partial\Gamma \cap \tilde{\Omega} \cap \{t > -r\} = \ell_1 \cup \ell_2$. Again, because of the maximum principle it is enough to show (2.5) on $\ell_1 \cup \ell_2$. Because the x -distance from any point on $\ell_1 \cup \ell_2$ to $\partial\tilde{\Omega}$ is at most $B\sqrt{\delta} + B\sqrt{R}/2 < B\sqrt{R}$, we deduce from Lemma 7 that

$$\tilde{\omega}^{(x, t)}(\Psi(0, R)) > c \quad \text{on } \ell_1 \cup \ell_2.$$

Because $r < R\varepsilon^{2+2\Lambda_2/\Lambda_1} < \delta\varepsilon^{2\Lambda_2/\Lambda_1}$, we deduce from Lemma 9 that on ℓ_1 ,

$$(5.2) \quad \tilde{\omega}^{(x, \delta)}(\Psi(0, r)) < c\varepsilon.$$

Since $B\sqrt{R}/2 > (20\sqrt{\Lambda_2 \log 1/\varepsilon + B})\sqrt{\delta}$, we see from Lemma 8 that (5.2) holds also on ℓ_2 . This completes the proof of Proposition 3.

6. DIRICHLET REGULAR DOMAINS AND THE MAXIMUM PRINCIPLE

We consider the Dirichlet problem with continuous, compactly supported boundary data.

Theorem 2. Let L be a parabolic operator and D be a domain in $\mathbf{R}^2$. A point $(x_0, t_0) \in \partial D$ is a regular boundary point for L if

(i) $\mathbf{R}^2 \setminus D$ contains a line segment $\{(x_0, s): t_0 - \delta \leq s \leq t_0\}$, or (ii) $\mathbf{R}^2 \setminus D$ contains a region $\{(x, t): t \leq t_0 \text{ and } -d \leq x - x_0 \leq -b\sqrt{-(t - t_0)}\}$ for some $b, d > 0$.

In particular, domains $\{t > \tau(x)\}$ with τ continuous on $\mathbf{R}$, and domains $\{x > \chi(t)\}$ with $\chi \in \text{Lip } \frac{1}{2}$ on $\mathbf{R}$, are Dirichlet regular for L .

Proof. For the heat equation, regularity is proved in [2, Theorem 5] and [7, pp. 389–393] for the cases (i) and (ii) respectively.

Assume, for simplicity, that $(x_0, t_0) = (0, 0)$. In the case (i), we let $v(x, t) = \int_{-\delta}^0 F(x, t-s) ds$, where F is defined in (3.1), and note that v attains a strict absolute maximum at $(0, 0)$. In view of Lemma D, $Lv \geq 0$ on $\mathbf{R}^2 \setminus \{(0, s): -\delta \leq s \leq 0\}$. Therefore $v(0, 0) - v(x, t)$ is a barrier at $(0, 0)$. Thus $(0, 0)$ is a regular boundary point for L [8].

In the case (ii), we follow closely the ideas in [7]. First we show that $(0, 0)$ is a regular boundary point of the domain $U \equiv \{(x, t): -1 < t < 0, -b < x/\sqrt{-t} < b\}$: let Λ_1 and Λ_2 be the constants in (0.2) associated with L , and let $\beta > 0$; for $(x, t) \in U$ with $-\log(-t) > (2\beta\Lambda_2/\Lambda_1)e^{b^2/4\Lambda_2}$ the function w defined by

$$(6.1) \quad w(x, t) = -(-\log(-t))^{-\beta-1} e^{-x^2/4\Lambda_2 t} + \frac{\Lambda_1}{2\beta\Lambda_2} (-\log(-t))^{-\beta}$$

is a barrier at $(0,0)$ for the domain U ($w > 0, Lw \leq 0$ and $w \rightarrow 0$ as $(x, t) \rightarrow (0, 0)$ in U). To deduce that $(0, 0)$ is a regular boundary point for D , we follow step by step the proofs of [7, Theorems 2 and 3], where a similar statement for the heat equation is obtained. The proofs can be adapted to the parabolic operator L in the obvious way, once the following property is proved.

Proposition 4. *Given a parabolic operator L and a domain $U \equiv \{(x, t): -1 < t < 0, -b < x/\sqrt{-t} < b\}$, let v be a bounded function satisfying $Lv \geq 0$ on U and having boundary values 0 on $\{x = -b\sqrt{-t}, t < 0\}$, and 1 on $\{x = b\sqrt{-t}, t < 0\}$. Then $\limsup_{t \rightarrow 0} v(0, t) < 1$.*

Proof. Let

$$(6.2) \quad \phi(\rho) = \int_0^\rho e^{s^2/4} ds,$$

and define for $t < 0$,

$$(6.3) \quad u(x, t) = \begin{cases} \phi\left(\frac{x}{\sqrt{-\Lambda_2 t}}\right), & x \geq 0, \\ \sqrt{\frac{\Lambda_1}{\Lambda_2}} \phi\left(\frac{x}{\sqrt{-\Lambda_1 t}}\right), & x < 0. \end{cases}$$

Straightforward differentiation shows that $Lu \leq 0$. Let, for $t < 0$,

$$f(x, t) = \frac{u(x, t) - \sqrt{\frac{\Lambda_1}{\Lambda_2}} \phi\left(\frac{-b}{\sqrt{\Lambda_1}}\right)}{\phi\left(\frac{b}{\sqrt{\Lambda_2}}\right) - \sqrt{\frac{\Lambda_1}{\Lambda_2}} \phi\left(\frac{-b}{\sqrt{\Lambda_1}}\right)}.$$

It is clear that $Lf \leq 0$, $f = 0$ on $x = -b\sqrt{-t}$ and $f = 1$ on $x = b\sqrt{-t}$, and $0 < f < 1$ in $-b < x/\sqrt{-t} < b$. Moreover, f is a constant on the line $\{x = 0, t < 0\}$. It follows from the maximum principle for bounded domains and the fact that $(0, 0)$ is regular for U , that $\limsup_{t \rightarrow 0} v(0, t) \leq f(0, -1) < 1$.

Theorem 3. *Let L be a parabolic operator and u be a subsolution on $D = \{t > \tau(x)\}$ with τ continuous on $\mathbf{R}$. Suppose that $u \leq 1$ on D and that $\limsup u \leq 0$ as (x, t) approaches any point on ∂D . Then $u \leq 0$ in D .*

Proof. Fix $(x_0, t_0) \in D$. Let $(y, \tau(x_0)) \in D$, $M > 5|y - x_0|$, and $D_M = D \cap \{|x - x_0| < M\}$. Note that for sufficiently large $T > 0$, $D_M \subseteq V \equiv \{|x - x_0| < M, t - t_0 > -T\} \setminus \{x = x_0, t \leq \tau(x_0)\}$, and that

$$u(y, \tau(x_0)) \leq \omega_{V, \tau(x_0)}^{(y, \tau(x_0))}(\{|x - x_0| = M\} \cap \partial V)$$

by the maximum principle for bounded domains. The right hand side is bounded above by $|y - x_0|(M - |y - x_0|)^{-1}$ in view of Lemma C. Letting $M \rightarrow \infty$, we have $u(y, \tau(x_0)) \leq 0$. Thus $u \leq 0$ on $D \cap \{t = \tau(x_0)\}$. We deduce from Lemma A and the maximum principle for bounded domains that $u(x_0, t_0) \leq 0$.

Theorem 4. *Let L be a parabolic operator and u be a subsolution on $D = \{x > \chi(t)\}$ with $\chi \in \text{Lip} \frac{1}{2}$ on $\mathbf{R}$. Suppose that $u \leq 1$ on D and that $\limsup u \leq 0$ at every boundary point of D . Then $u \leq 0$ in D .*

Proof. Choose $B > 1$, so that

$$|\chi(t) - \chi(s)| \leq B|t - s|^{1/2} \quad \text{for all } t, s \in R.$$

Given $0 < \varepsilon < \frac{1}{2}$ and $(x_0, t_0) \in D$, we need to show that $u(x_0, t_0) < \varepsilon$. The fact that every boundary point of D is Dirichlet regular is implicitly used in the proof.

Choose and fix a point Q on ∂D with t -coordinate $> t_0$, so that after a translation, $Q = (0, 0)$, $t_0 < 0$ and $|x_0| < \beta\sqrt{-t_0}$.

Choose and fix a number $k > 2/\varepsilon$ which satisfies

$$(6.4) \quad \left[\phi\left(\frac{B}{\sqrt{\Lambda_2}}\right) - \sqrt{\frac{\Lambda_1}{\Lambda_2}} \phi\left(\frac{-B}{\sqrt{\Lambda_1}}\right) \right] \left[\phi\left(\frac{kB}{\sqrt{\Lambda_2}}\right) - \sqrt{\frac{\Lambda_1}{\Lambda_2}} \phi\left(\frac{-B}{\sqrt{\Lambda_1}}\right) \right]^{-1} < \varepsilon,$$

where ϕ is the function in (6.2). Let

$$U = \{t < 0, -B\sqrt{-t} < x < kB\sqrt{-t}\},$$

and define for $t < 0$,

$$v(x, t) = \frac{u(x, t) - \sqrt{\frac{\Lambda_1}{\Lambda_2}} \phi\left(\frac{-B}{\sqrt{\Lambda_1}}\right)}{\phi\left(\frac{kB}{\sqrt{\Lambda_2}}\right) - \sqrt{\frac{\Lambda_1}{\Lambda_2}} \phi\left(\frac{-B}{\sqrt{\Lambda_1}}\right)},$$

where u is the function in (6.3). Clearly $Lv \leq 0$, $v = 1$ on $S \equiv \{x = kB\sqrt{-t}, t < 0\}$, and $v = 0$ on $\{x = -B\sqrt{-t}, t < 0\}$; moreover, we see from (6.4) that $\sup_{-B\sqrt{-t} < x < kB\sqrt{-t}} v(x, t) < \varepsilon$, in particular $v(x_0, t_0) < \varepsilon$.

Let $T > -10t_0$ and $U_T = U \cap \{t > -T\}$. It follows from the maximum principle for bounded domains that

$$u(x_0, t_0) < v(x_0, t_0) + \omega_{U_T}^{(x_0, t_0)}(\partial U_T \cap \{t = -T\}).$$

After a change of scales, we obtain for sufficiently large T ,

$$\omega_{U_T}^{(x_0, t_0)}(\partial U_T \cap \{t = -T\}) \leq w\left(\frac{x_0}{\sqrt{T}}, \frac{t_0}{T}\right),$$

where w is the function defined in (6.1). Letting $t \rightarrow \infty$, we conclude that $u(x_0, t_0) < \varepsilon$. This concludes the proof.

Standard deductions give the maximum principle for bounded solutions of $Lu = 0$, and the uniqueness theorem for the Dirichlet problem in domains described in Theorems 3 and 4.

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