WEAKLY ALMOST PERIODIC ELEMENTS IN $L_\infty(G)$ OF A LOCALLY COMPACT GROUP

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ABSTRACT. Let G be a locally compact abelian group with dual group $\hat{G}$ acting on $L_\infty(G)$ by pointwise multiplication. We show that if $L_\infty(G)$ contains a nonzero element f such that $O(f) = \{\chi \cdot f : \chi \in \hat{G}\}$ is relatively compact in the weak (or norm) topology of $L_\infty(G)$, then G is discrete. In this case $O(f)$ is relatively compact in the weak or norm topology of $L_\infty(G)$ if and only if f vanishes at infinity. A related result when G acts on the von Neumann algebra $VN(G)$ is also determined.

1. INTRODUCTION

Let G be a locally compact abelian group and $\hat{G}$ be its dual group. Then $\hat{G}$ acts naturally on $L_\infty(G)$ by pointwise multiplication. Let $B(G)$ be the Fourier Stieltjes algebra of G . Then $B(G)$ is a subalgebra of $L_\infty(G)$ invariant under the action of $\hat{G}$. A well-known result of R. R. Goldberg and B. A. Simon [7, Theorem B] implies that if $f \in B(G)$, then f vanishes at infinity if and only if $\|\chi \cdot f - f\|_\infty \rightarrow 0$ as $\chi \rightarrow 1$. In this paper we prove, among other things, that if $L_\infty(G)$ contains a nonzero element f such that $O(f) = \{\chi \cdot f : \chi \in \hat{G}\}$ is relatively compact in the weak topology of $L_\infty(G)$, then G is discrete. In this case $O(f)$ is relatively compact in the weak (or norm) topology of $L_\infty(G)$ if and only if f vanishes at infinity. A related result is also obtained when G acts on the von Neumann algebra $VN(G)$ generated by left translations on $L_2(G)$.

2. PRELIMINARIES AND SOME NOTATIONS

Let G be a locally compact group (not necessarily abelian) with left Haar measure λ , $N(G)$ the closed subspace of $BM(G)$ consisting of all locally null functions on G , and $L_\infty(G)$, the quotient Banach space $BM(G)/N(G)$ with the quotient norm $\|\cdot\|_\infty$ (i.e. the essential sup norm). Given $f \in BM(G)$,

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$\bar{f}$ will denote its equivalence class in $L_\infty(G)$. When there is no confusion, we shall write f instead of $\bar{f}$.

Let $L_1(G)$ denote the Banach space of integrable complex-valued functions φ on G with L_1 -norm $\|\varphi\|_1 = \int |\varphi(s)| d\lambda(s)$ and let $P_1(G)$ denote all $\varphi \in L_1(G)$ such that $\|\varphi\|_1 = 1$ and $\varphi \geq 0$. If $\varphi \in L_1(G)$ and $f \in L_\infty(G)$, then the function $(\varphi * f)(s) = \int \varphi(t) f(t^{-1}s) d\lambda(t)$ defined for each $s \in G$ is bounded and right uniformly continuous (see [9, p. 295]). Also, for $a \in G$, let $\delta(a)$ denote the point evaluation functional on $CB(G)$ defined by $\delta(a)(f) = f(a)$, $f \in CB(G)$.

If A is a subset of a locally convex space E with topology τ , then the convex hull of A will be denoted by coA and the τ -closed convex hull of A in E will be denoted by $\tau\overline{co}(A)$ or $\overline{co}(A)$ (when there is no confusion). The τ -closure of A will be written as τA^- .

A left Banach G -module X is a Banach space X which is a left G -module such that

- (i) $e \cdot x = x$ for all $x \in X$, where e is the identity of G .
- (ii) $\|s \cdot x\| \leq \|x\|$ for all $x \in X$ and $s \in G$.
- (iii) For all $x \in X$, the map $s \rightarrow s \cdot x$ is continuous from G into X .

In this case, we can define for each $f \in X^*$, $s \in G$, the element $f \cdot s \in X^*$ by

$$\langle f \cdot s, x \rangle = \langle f, s \cdot x \rangle, \quad x \in X.$$

Define also $\langle f \cdot \mu, x \rangle = \int \langle f, s \cdot x \rangle d\mu(s)$, $\mu \in M(G)$, $f \in X^*$, $x \in X$, where $M(G)$ is the space of (complex, bounded) Radon measures on G . Then $f \cdot \mu \in X^*$, $f \cdot \mu = f \cdot s$ if $\mu = \delta(s)$ and $(f \cdot \mu_1) \cdot \mu_2 = f \cdot (\mu_1 * \mu_2)$ for $\mu_1, \mu_2 \in M(G)$.

The following lemma can be proved by elementary arguments similar to that of Lemma 2 in [14, p. 151]; we safely omit the details.

Lemma 2.1. *Let G be a locally compact group and X be a left Banach G -module. If $f \in X^*$, then $w^*\overline{co}\{f \cdot s : s \in G\} = w^*\{f \cdot \varphi : \varphi \in P_1(G)\}^-$.*

Corollary 2.2. *Let G be a locally compact group and X be a left Banach G -module. Let $f \in X^*$ be such that $\{f \cdot s : s \in G\}$ is relatively compact in the weak topology of X^* ; then there exists a sequence $\varphi_n \in P_1(G)$ such that $\|f - f \cdot \varphi_n\| \rightarrow 0$.*

Proof. Let $K = \overline{co}\{f \cdot s : s \in G\}$ (norm or weak closed convex hull). Since (K, weak) is compact and (K, weak^*) is Hausdorff, the weak and weak* topologies agree on K . By Lemma 2.1, $K = w^*\{f \cdot \varphi : \varphi \in P_1(G)\}^- = \{f \cdot \varphi : \varphi \in P_1(G)\}^-$ (weak or norm closure). Since $f \in K$, the conclusion follows. $\square$

Remark. Since $L_1(G)$ has a bounded approximate identity, an application of Cohen's Factorization theorem implies that if $f \in X^*$ and $\{f \cdot s : s \in G\}$ is relatively compact in the weak topology of X^* , then $f = h \cdot \varphi$ for some $h \in X^*$ and $\varphi \in L_1(G)$. However, this will not be needed in our applications.

Corollary 2.3. *Let G be a compact group and X be a left Banach G -module. Let $f \in X^*$. The following statements are equivalent:*

- (a) $\{f \cdot s : s \in G\}$ is relatively compact in the weak topology of X^* .
- (b) $\{f \cdot s : s \in G\}$ is relatively compact in the norm topology of X^* .
- (c) There exists a sequence $\varphi_n \in P_1(G)$ such that $\|f - f \cdot \varphi_n\| \rightarrow 0$.
- (d) The map $s \rightarrow f \cdot s$ from G into $(X^*, \|\cdot\|)$ is continuous.

Proof. This follows from Corollary 2.2 and the observation that if $h = f \cdot \varphi$, $\varphi \in P_1(G)$, then the mapping $s \rightarrow h \cdot s$ from G into $(X^*, \|\cdot\|)$ is continuous. $\square$

3. THE MAIN THEOREM

Let G be a locally compact abelian group. Then $\hat{G}$ acts continuously on $L_1(G)$ by pointwise multiplication. The “dual” of this action of $\hat{G}$ on $L_\infty(G) = L_1(G)^*$ is also defined by pointwise multiplication.

Theorem 3.1. *Let G be a locally compact abelian group. If there exists a nonzero $f \in L_\infty(G)$ such that $O(f) = \{\chi \cdot f, \chi \in \hat{G}\}$ is relatively compact in the weak topology of $L_\infty(G)$, then G is discrete. In this case, $O(f)$ is relatively compact in the weak (or norm) topology of $L_\infty(G)$ if and only if f vanishes at infinity.*

Proof. Suppose G is nondiscrete. Let $\rho: BM(G) \rightarrow BM(G)$ be a lifting which commutes with translations (see [11] or [10, p. 348]) and $\bar{\rho}: L_\infty(G) \rightarrow BM(G)$ be defined by $\bar{\rho}(\bar{f}) = \rho(f)$, $\bar{f} \in L_\infty(G)$. Assume that $O(\bar{f})$ is relatively compact in the weak topology of $L_\infty(G)$ and $\bar{f} \neq 0$. Since $\bar{\rho}(\chi \cdot \bar{f}) = \chi \cdot \bar{\rho}(\bar{f})$ for all $\chi \in \hat{G}$, it follows that $\{\chi \cdot \bar{\rho}(\bar{f}) : \chi \in \hat{G}\}$ is relatively compact in the weak topology of $BM(G)$. Let $h = \bar{\rho}(\bar{f})$. If $h(e) = \alpha \neq 0$, where e is the identity of G , then $(\chi \cdot h)(e) = \chi(e)h(e) = \alpha$ for all $\chi \in \hat{G}$. Hence $k(e) = \alpha$ for all k in $\Delta = \overline{c\bar{o}}\{\chi \cdot h : \chi \in \hat{G}\}$. Consider the pointwise, hence weakly continuous, affine action of $\hat{G}$ on the weakly compact set Δ defined by pointwise multiplication. By the Markov-Kakutani fixed point theorem [5, p. 456], there exists $k_0 \in \Delta$ such that $\chi \cdot k_0 = k_0$ for all $\chi \in \hat{G}$ and $k_0 \neq 0$. Since $\overline{c\bar{o}}O(\bar{f})$ is weakly compact, $\bar{\rho}$ maps $\overline{c\bar{o}}O(\bar{f})$ onto Δ . Hence there exists $\bar{k} \in L_\infty(G)$ such that $\bar{\rho}(\bar{k}) = k_0$. Now if $s \neq e$, $s \in G$, there exists $\chi \in \hat{G}$ such that $\chi(s) \neq 1$. But

$$\chi(s)k_0(s) = k_0(s).$$

Hence $k_0(s) = 0$. So support of $k_0 = \{e\}$, which has measure zero when G is not discrete, i.e. $\bar{k}_0 = 0$. Hence $\bar{k} = 0$. Since $\|\bar{\rho}(\bar{k})\| = \|\bar{k}\|_\infty$ (see [10, p. 348]), it follows that $k_0 \equiv 0$, which is impossible. Hence $h(e) = 0$.

Finally, for each $s \in G$, consider $h_1 = \bar{\rho}(\bar{f}_s)$ with $h_1(e) = \bar{\rho}(\bar{f}_s)(e) = h(s)$, where $\bar{f}_s$ denotes the translation of $\bar{f}$ by s . Also, if $\chi \in \hat{G}$, then

$$\chi \cdot \bar{f}_s = [\chi(s^{-1})\chi \cdot \bar{f}]_s.$$

Since $O(\bar{f})$ is relatively compact in the weak topology of $L_\infty(G)$, and $\{\chi(s^{-1}): \chi \in \hat{G}\}$ is a bounded set, $\{\chi(s)\chi \cdot \bar{f}: \chi \in \hat{G}\} = K$ is relatively compact in the weak topology of $L_\infty(G)$. Hence $O(\bar{f}_s)$, which is a translate of K , is also relatively compact in the weak topology of $L_\infty(G)$. The above arguments show that $h(s) = h_1(e) = 0 \forall s \in G$. This contradicts the fact that $\bar{f} \neq 0$. Hence G must be discrete.

If G is discrete, then $\hat{G}$ is compact. Since the action of $\hat{G}$ on $L_1(G)$ (defined by pointwise multiplication) is continuous, Corollary 2.3 applied to the compact group $\hat{G}$ implies that $O(f)$ is relatively compact in the weak (or norm) topology of $L_\infty(G)$ iff $\|\chi \cdot f - f\|_\infty \rightarrow 0$ as $\chi \rightarrow 1$. But this is equivalent to the assertion that f vanishes at infinity by Theorem 4.4 in [8]. $\square$

4. NONABELIAN GROUPS

Let G be a locally compact group and $A(G)$ denote Fourier algebra of G , i.e. all functions φ on G of the form

$$\varphi(x) = \langle \ell_x h, k \rangle, \quad x \in G, \quad h, k \in L_2(G)$$

where $\ell_x h(y) = h(x^{-1}y)$, $y \in G$. Then $A(G)$ may be identified with the predual of the von Neumann algebra $VN(G)$ on $L_2(G)$ generated by left translations $\{\ell_x: x \in G\}$. $A(G)$ with the norm

$$\|\varphi\| = \sup\{|\varphi(T)|: T \in VN(G) \text{ and } \|T\| \leq 1\}$$

and pointwise multiplication is a semisimple commutative Banach algebra with spectrum G . Furthermore if G is commutative, then $A(G) \simeq L(\hat{G})$ and $VN(G) \simeq L_\infty(\hat{G})$ (see Eymard [6] for more details). It follows from this identification and Theorem 3.1 that if G is commutative and there is a nonzero operator $T \in VN(G)$ such that $\{\ell_x T: x \in G\}$ is relatively compact in the weak topology of $VN(G)$, then G is compact. We do not know if the corresponding result is true for $VN(G)$ for general G . However we have the following weaker generalization of Theorem 3.1:

Proposition 4.1. *Let G be a locally compact group. If there exists a nonzero operator $T \in VN(G)$ such that $O(T) = \{\ell_g T: g \in G\}$ is relatively compact in the norm topology of $VN(G)$, then G is compact. In this case, $O(T)$ is relatively compact in the weak (or norm) topology of $VN(G)$ if and only if T is in $C_p^*(G)$, the C^* -algebra in $\mathcal{B}(L_2(G))$ generated by convolution by functions in $L_1(G)$.*

Proof. If $T \in VN(G)$ is a nonzero operator such that $\{\ell_x T: x \in G\}$ is relatively compact in the norm topology of $VN(G)$, let $h \in L_2(G)$ be such that $T(h) = k \neq 0$. Then $K = \overline{\text{co}}\{\ell_x k: x \in G\}$ is a nontrivial compact convex left invariant subset of $L_2(G)$. So by [12, Theorem 4.5], G must be compact. Also, since left translation acts continuously on $A(G)$, the last statement is a direct consequence of Corollary 2.3 and Theorem 4.4 of [8]. $\square$

5. SOME ADDITIONAL REMARKS

Let G be a locally compact group. A function f in $BM(G)$ is almost periodic if $\{\ell_x f: x \in G\}$ is relatively compact in the sup norm topology of $BM(G)$. Corollary 2.2 can be used to obtain an elementary proof of the following well-known result of von Neumann (see [3]).

Proposition 5.1. *Let G be a locally compact group and $f \in BM(G)$. If f is almost periodic, then f is continuous on G .*

Proof. If f is almost periodic, then $\{\bar{s}f: s \in G\}$ is relatively compact in the norm topology of $L_\infty(G)$ where $(_s f)(t) = f(st)$, $t \in G$. Hence by Corollary 2.2, $f = h$ for some $h \in CB(G)$ locally a.e. We must show that $f \equiv h$. We may assume without loss of generality, that $h = 0$. Let $\delta = \|f\|_u/2 > 0$. We shall construct a sequence a_n in G such that $\|_{a_i} f -_{a_j} f\|_u > \delta$ for any $i \neq j$. Let $a_1 \in G$. Suppose $a_1, \dots, a_n$ have been chosen such that $\|_{a_i} f -_{a_j} f\|_u > \delta$ for any $i \leq i \neq j \leq n$. For each $1 \leq i \leq n$, let $x_i \in G$ be such that $|f(a_i x_i)| > \delta$. Choose $a_{n+1} \in G$ such that $f(a_{n+1} x_i) = 0$ for all $i = 1, \dots, n$ (this is possible since the support of f is locally null). Consequently the sequence $(_{a_n} f)$ of translates f cannot have any convergent subsequence. This contradicts the assumption. $\square$

Corollary 2.2 can also be applied to give a short proof of the following result which was proved in Civin and Yood [2] for locally compact abelian groups G and in Young [16] for all G .

Proposition 5.2. *If $L_1(G)$ is Arens regular, then G is finite.*

Proof. As in the proof of Corollary 5 in [15], if $L_1(G)$ is Arens regular, then for each $f \in L_\infty(G)$, the set $\{\varphi * f: \varphi \in P_l(G)\}$ is relatively compact in the weak topology of $L_\infty(G)$ [4, p. 312]. By Corollary 2.2 (and its proof), $L_\infty(G) = CB(G)$. So G must be discrete. Using Corollary 2.2 again, $CB(G) = W(G)$, the space of weakly almost periodic function on G . So G must be compact ([1, p. 68]) and hence finite. $\square$

Civin and Yood's result [2] can also be extended to the Fourier algebra $A(G)$ when G is amenable.

Proposition 5.3. *Let G be an amenable locally compact group. If $A(G)$ is Arens regular, then G is finite.*

Proof. Since $A(G)$ is commutative, $VN(G)^*$ (the second conjugate algebra of $A(G)$) is also commutative when $A(G)$ is Arens regular [4, p. 311]. Since G is amenable, G must be finite by Proposition 3.3 in [13]. $\square$

We do not know if Proposition 5.3 still remains valid when G is nonamenable.

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