TWO-DIMENSIONAL REPRESENTATIONS OF GROUPS WITH PROPERTY FA

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ABSTRACT. We show that the two-dimensional representations of groups having property FA have finite image, if they always have compact closure.

In this note we generalize a theorem of Zimmer [Z; Theorem 7] that concerned groups having Kazhdan's property T to the class of groups having property FA. It is known that a group having property T also has property FA of Serre, [W₁]. Where the results of [Z] relied on information concerning the decay of matrix entries of unitary representations of the given group, we shall instead use actions on trees.

Theorem. *If Γ is a finitely generated group of type FA such that every 2-dimensional representation*

$$\rho: \Gamma \mapsto \mathrm{GL}_2(\mathbb{C})$$

has image with compact closure, then $\rho(\Gamma)$ is finite for every two-dimensional representation.

The proof of the following corollary follows easily from the theorem.

Corollary. *A finitely generated subgroup Γ of $\mathrm{GL}_2(\mathbb{C})$ is finite if and only if (1) Γ is of type FA and (2) every 2-dimensional representation*

$$\rho: \Gamma \mapsto \mathrm{GL}_2(\mathbb{C})$$

has image with compact closure.

Proof of theorem. According to the reformulation and extension of Serre's FA property [B; Theorem 1] the subgroup $\rho(\Gamma)$ of $\mathrm{GL}_2(\mathbb{C})$ is either: (a) a free product with amalgamation; (b) has a nontrivial homomorphism to $\mathbb{Z}$; (c) conjugate to an upper triangular group with roots of unity on the diagonal; or (d) conjugate to a subgroup of $\mathrm{GL}_2(A)$ for a ring of integers A .

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By property FA [S; Chapter 6], cases (a) and (b) cannot occur. In case (c) $\rho(\Gamma)$ has a subgroup of finite index which is unipotent; so $\rho(\Gamma)_u$ is trivial and $\rho(\Gamma)$ is finite. In case (d) since $\rho(\Gamma)$ has compact closure all of the eigenvalues have absolute value 1. Furthermore, by use of the characteristic polynomial, we find that the eigenvalues of any element of $\rho(\Gamma)$ are in some integral quadratic extension B of A ; it follows from the compactness assumption on $\rho(\Gamma)$ that the eigenvalues must have absolute value one for every embedding of B in $\mathbf{R}$ or $\mathbf{C}$. Thus, it follows from the Dirichlet Unit Theorem that the eigenvalues of any given element are roots of unity. Moreover, for all elements of $\rho(\Gamma)$, these roots of unity have degree bounded by twice the dimension of the quotient number field of A and hence the eigenvalues are finite in number. Now, Greenberg's Lemma [W_2 ; Lemma 4.5] asserts that a finitely generated linear group has a subgroup of finite index with traces (other than 2) not belonging to some given finite set; since there are finitely many roots of unity to avoid, and therefore finitely many traces to avoid, $\rho(\Gamma)$ has a subgroup of finite index whose eigenvalues are all 1 and therefore $\rho(\Gamma)$ is finite.

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