THE NONEXISTENCE OF EXPANSIVE HOMEOMORPHISMS OF 1-DIMENSIONAL COMPACT ANRS

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ABSTRACT. It is well known that if X is an arc or a circle, then there is no expansive homeomorphism on X (see [2] and [3]). In this note, we show that if X is a Peano continuum which has a neighborhood M such that $\text{cl}(M)$ is a 1-dimensional AR, then there is no expansive homeomorphism on X . In particular, no 1-dimensional compact ANR admits an expansive homeomorphism.

1. INTRODUCTION

Let X be a compact metric space with metric d . A homeomorphism f of X is *expansive* if there exists $c > 0$ (called an *expansivity constant* for f) such that $d(f^n(x), f^n(y)) \leq c$ for all integer n implies $x = y$. It is well known that the 2-adic solenoid and the 2-torus admit expansive homeomorphisms (see [6] and [7]). Also, Bryant [2] proved that there is no expansive homeomorphism on an arc. Jakobsen and Utz proved that there is no expansive homeomorphism on a circle [3].

By using those results, Kawamura showed that if X is a Peano continuum and X contains a free arc, then X does not admit an expansive homeomorphism [4]. Naturally, the following problem arises: If X is a 1-dimensional compact AR (or 1-dimensional compact ANR), is it true that X does not admit an expansive homeomorphism? It is well known that there are many kinds of 1-dimensional compact ARs (or 1-dimensional compact ANRs) which contain no free arcs. In this note, we prove that if X is a Peano continuum which has a neighborhood M such that $\text{cl}(M)$ is a 1-dimensional AR, then there exists no expansive homeomorphism on X . In particular, no 1-dimensional compact ANR admits an expansive homeomorphism. The proof is different from the proofs of Bryant, Jakobsen and Utz ([2] and [3]).

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2. SELF-HOMEOMORPHISMS OF DENDRITES

In this section, we prove that if X is a dendrite (=1-dimensional compact AR), then there exists no expansive homeomorphism on X . To prove this result, we use an idea of Mañé [5].

2.1. Lemma [1, (13.5)]. *A continuum X is a 1-dimensional AR if and only if X is a nondegenerate Peano continuum and X contains no circle.*

2.2. Lemma [5, Lemma 2.I]. *Let $f: X \rightarrow X$ be an expansive homeomorphism of a compact metric space X and let $c > 0$ be an expansivity constant for f and $0 < \varepsilon < c/2$. Then there exists $\delta > 0$ such that if $x, y \in X$, $d(x, y) < \delta$, and $\varepsilon \leq \sup\{d(f^j(x), f^j(y)) \mid 0 \leq j \leq n\} \leq 2\varepsilon$ for some $n \geq 0$, then $d(f^n(x), f^n(y)) \geq \delta$.*

2.3. Theorem. *Let X be a 1-dimensional compact AR. If $f: X \rightarrow X$ is a homeomorphism, then f is not expansive.*

Proof. Suppose, on the contrary, that f is expansive. Let $\delta > 0$ be as in 2.2. Since X is locally connected, there are subcontinua $X_1, X_2, \dots, X_m$ of X such that $X = \cup X_i$ and $\text{diam } X_i < \delta$ for each i . Let $X_{i,j}^n = f^n(X_i) \cap f^{-n}(X_j)$. Since $f^n(X_i)$ and $f^{-n}(X_j)$ are subcontinua in X and X contains no circle, $X_{i,j}^n = f^n(X_i) \cap f^{-n}(X_j)$ is connected (it may be empty). By the proof of [5, p. 318–319], we can conclude that $\lim_{n \rightarrow \infty} (\sup_{i,j} \text{diam } X_{i,j}^n) = 0$. Note that $X = \cup \{X_{i,j}^n \mid 1 \leq i, j \leq m\}$. Since $\text{diam } X \leq \sum_{i,j=1}^m \text{diam } X_{i,j}^n$, we can conclude that X is degenerate, which is a contradiction.

3. SELF-HOMEOMORPHISMS OF CERTAIN PEANO CONTINUA

In this section, we prove that if X is a Peano continuum and X contains a neighborhood M such that $\text{cl}(M)$ is a 1-dimensional AR, X does not admit an expansive homeomorphism. In particular, no 1-dimensional compact ANR admits an expansive homeomorphism.

3.1. Lemma [1, (13.6)]. *A continuum X is a 1-dimensional ANR if and only if X is a Peano continuum of dimension 1 and X does not contain infinite circles.*

3.2. Theorem. *If X is a Peano continuum which has a neighborhood M such that $\text{cl}(M)$ is a 1-dimensional AR, then X does not admit an expansive homeomorphism. In particular, no 1-dimensional compact ANR admits an expansive homeomorphism.*

Proof. By 2.1, $\text{cl}(M)$ is a dendrite. Choose nonempty open sets U_1, U_2 in M such that $\text{cl } U_1 \subset U_2 \subset \text{cl } U_2 \subset M$. Let Y be a component of $\text{cl } U_1$ such that $U_1 \cap Y \neq \emptyset$. Note that Y is a dendrite. Suppose, on the contrary, that there is an expansive homeomorphism $f: X \rightarrow X$. Let $\delta > 0$ be as in 2.2 and let $X_1, X_2, \dots, X_m$ be subcontinua of X such that $X = \bigcup_{i=1}^m X_i$ and $\text{diam } X_i < \delta$

for each i . Note that there is a natural number $p > 0$ such that if Z is a subcontinuum of X , then the number of components of $Z \cap Y$ is less than p . Suppose, on the contrary, that there is a sequence $\{Z_n\}$ of subcontinua of X such that $|\{C \mid C \text{ is a component of } Y \cap Z_n\}| \geq n$, where $|A|$ denotes the cardinal number of a set A . Since M does not contain a circle, we can see that $|\{C \mid C \text{ is a component of } (\text{cl } U_2 - U_1) \text{ with } C \cap \text{cl } U_1 \neq \emptyset \neq C \cap X - U_2\}| = \infty$. This implies that X is not locally connected, which is a contradiction.

Now, let $X_{i,j}^n = f^n(X_i) \cap f^{-n}(X_j) \cap Y$. Since Y contains no circle, we can conclude that the number of components of $X_{i,j}^n$ is less than p^2 . By the proof of [5, p. 318–319], $\lim_{n \rightarrow \infty} \text{diam } C_{i,j}^{n,k} = 0$, where $C_{i,j}^{n,k}$ is a component of $X_{i,j}^n$. Since $Y = \bigcup C_{i,j}^{n,k}$, we can conclude that $\text{diam } Y = 0$. This is a contradiction.

Question 1. Is it true that Menger's universal curve admits an expansive homeomorphism?

Question 2. If X is a dendroid (= pathwise connected treelike continuum) or a treelike continuum, is it true that X does not admit an expansive homeomorphism?

Added in proof. Recently, the author proved that there are no expansive homeomorphisms on hereditarily decomposable treelike (or circlelike) continua.

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