L^∞ -BMO BOUNDEDNESS FOR A SINGULAR INTEGRAL OPERATOR

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ABSTRACT. If $K(x) = \Omega(x)/|x|^n$ is a Calderón-Zygmund kernel and $b(|x|)$ is a bounded radial function, we find conditions on b such that the singular operator whose kernel is $b(x)K(x)$ is bounded from $L^\infty(R^n)$ to $BMO(R^n)$, or equivalently from $H^1(R^n)$ into $L^1(R^n)$.

Let $K(x) = \Omega(x)/|x|^n$ be a Calderón-Zygmund kernel, where

$$\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0,$$

and $\Omega(\lambda x) = \Omega(x)$ for $\lambda > 0$, $x \in R^n$. Let $H(x) = b(x)K(x)$, and define

$$Tf(x) = \lim_{\varepsilon \rightarrow 0} \int_{|y| > \varepsilon} H(y)f(x-y) dy = \mathbf{p} \cdot \mathbf{v} \cdot H * f(y).$$

In [5], we proved that if $\Omega \in L^q(S^{n-1})$ for some $1 < q \leq \infty$ and b is radial, then T is bounded on $L^p(R^n)$ for $1 < p < \infty$, $n \geq 2$. We recall that the space of locally integrable functions modulo constants via the norm

$$(0.1) \quad \|f\|_{BMO} = \sup_Q \frac{1}{|Q|} \int_Q |f - f_Q| < \infty,$$

is called $BMO(R^n)$, where the supremum is taken over all cubes Q whose sides are parallel to the Cartesian coordinates. It turns out that b must satisfy a certain degree of smoothness in order for T to be bounded from $L^\infty(R^n)$ to $BMO(R^n)$. Let $Q_r(x)$ be the cube of side length r centered at x , whose sides are parallel to the axes, and let

$$\Phi = \left\{ \phi \in L^\infty(R^n) : \text{suppt } \phi \subseteq Q_1(0), \|\phi\|_\infty \leq 1, \int \phi = 0 \right\},$$

$\phi_r(x) = r^{-n}\phi(x/r)$. We also assume K satisfies Hörmander's condition

$$(0.2) \quad \sup_{|y| > 0} \int_{|x| > 2|y|} |K(x+y) - K(x)| dx < c.$$

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Theorem 1. Suppose b is a bounded (not necessarily radial) and measurable function such that $H(x) = b(x)K(x)$ has a bounded Fourier transform, then the operator T maps $L^\infty(R^n)$ into $BMO(R^n)$ if and only if b satisfies

$$(0.3) \quad \sup_{\substack{r>0 \\ \phi \in \Phi}} \|K(x)(b * \phi_r(x))\|_{L^1(R^n \setminus Q_{2\sqrt{n}r})} < \infty.$$

Since for $\Omega \in L^q(S^{n-1})$, $1 < q \leq \infty$, $H(x)$ has a bounded Fourier transform for $n \geq 2$ [5], this leads to

Corollary. Let $\Omega \in L^q(S^{n-1})$, $1 < q \leq \infty$, $n \geq 2$, and $H(x) = b(|x|)K(x)$, where b is bounded. Then T maps L^∞ into BMO if and only if

$$(0.4) \quad \sup_{r>0, \phi \in \Phi} \|K(x)(b * \phi_r)(x)\|_{L^1(R^n \setminus Q_{2\sqrt{n}r})} < \infty.$$

Before we proceed with the proofs, we state the following result concerning weak type estimates; its proof is left to the reader.

We say a function $a(x)$ is a weighted 1-atom (with respect to a weight w) if

- (i) $\text{suppt } a \subseteq Q$, a cube Q ,
- (ii) $\int a(x) dx = 0$,
- (iii) $\int |a(x)|w(x) dx = 1$.

It is known that if w is an A_1 weight and Ω satisfies the Dini condition, then the operator S defined by $Sf(x) = \mathbf{p} \cdot \mathbf{v} \cdot K * f(x)$ satisfies the weighted weak type estimate

$$(0.5) \quad w(\{x \in R^n : |Sf(x)| > \lambda\}) \leq \frac{c}{\lambda} \|f\|_{L_w^1} \quad \text{for } \lambda > 0,$$

where c is independent of $f \in L_w^1(R^n)$ [3]. It turns out that in extending the above result to our operators, b will depend on w as well as K .

Theorem 2. Suppose $w \in A_1$, $b \in L^\infty(R^n)$, Ω satisfies the Dini condition and

$$(0.6) \quad \sup_Q \int_{R^n \setminus Q^*} |K(x - y_Q)(b * a)(x)|w(x) dx < \infty,$$

where supremum is taken over all weighted 1-atoms (with respect to w), $Q = \text{suppt } a$, $y_Q =$ center of Q , and Q^* is the cube centered at y_Q with twice as large a diameter as Q . If T is bounded on $L_w^2(R^n)$, then it is of weighted weak-type $(1, 1)$.

Let w be a weight and $L_w^\infty = \{f : \|f\|_{L_w^\infty} = \|f/w\|_\infty < \infty\}$. The weighted bounded mean oscillation space BMO_w is defined as

$$BMO_w = \{f \in L_{\text{loc}}^1(R^n) : \|f\|_{BMO_w} = \sup_Q \frac{1}{w(Q)} \int_Q |f - f_Q| < \infty\},$$

where $w(Q) = \int_Q w(x) dx$. Define

$$(1.1) \quad \|f\|_{BMO}^* = \sup_{\substack{\phi \in \Phi, \\ x \in R^n}} \sup_{r>0} \frac{|Q_r(x)|}{w(Q_r(x))} |(f * \phi_r)(x)|.$$

If $w \in A_p$ for some $1 \leq p < \infty$, then $\|f\|_{\text{BMO}_w}^* \approx \|f\|_{\text{BMO}_w}$, i.e.,

$$(1.2) \quad A' \|f\|_{\text{BMO}_w} \leq \|f\|_{\text{BMO}_w}^* \leq A \|f\|_{\text{BMO}_w},$$

where A and A' are constants independent of $f \in \text{BMO}_w$. We need the unweighted version of the following lemma. The more general statement is proved because it is of some interest in itself.

Lemma 1. *Let w be a weight and $K \in L^1(\mathbb{R}^n)$ be a kernel. If $Sf = K * (f/w)$ for $f \in L_w^\infty$, then S maps L_w^∞ continuously into BMO_w if and only if for each $r > 0$*

$$(1.3) \quad \sup_{\phi \in \Phi} \|K * \phi_r(x)\|_{L^1(\mathbb{R}^n)} \leq c A_r,$$

where $A_r = \inf_{x \in \mathbb{R}^n} w(Q_r(x))/|Q_r(x)|$ and the constant c is independent of $\|K\|_{L^1}$.

Proof of Lemma 1. Suppose S is bounded from L_w^∞ to BMO_w , then there is a constant c such that $\|K * (f/w)\|_{\text{BMO}_w} \leq c \|f/w\|_\infty$. By (1.2) for fixed $r > 0$, $x \in \mathbb{R}^n$,

$$\begin{aligned} \frac{|Q_r(x)|}{w(Q_r(x))} (K * (f/w) * \phi_r)(x) &\leq \|K * (f/w)\|_{\text{BMO}_w}^* \\ &\leq A \|K * (f/w)\|_{\text{BMO}_w} \\ &\leq c \|f/w\|. \end{aligned}$$

Thus

$$\frac{|Q_r(x)|}{w(Q_r(x))} \int (K * \phi_r)(y) (f/w)(x - y) dy \leq c A \|f/w\|_\infty.$$

For $x \in \mathbb{R}^n$, choose f such that $(f/w)(x - y) = \text{sgn} (K * \phi_r)(y)$, hence

$$\frac{|Q_r(x)|}{w(Q_r(x))} \|K * \phi_r\|_{L^1(\mathbb{R}^n)} \leq c A$$

or

$$\sup_{\phi \in \Phi} \|K * \phi_r\|_{L^1(\mathbb{R}^n)} \leq c \inf_{x \in \mathbb{R}^n} \frac{w(Q_r(x))}{|Q_r(x)|} = c A_r.$$

Conversely, suppose (1.3) holds. For each $\phi \in \Phi$ and each cube $Q = Q_r(x)$,

$$\frac{|Q|}{w(Q)} \int |(K * \phi_r)(y) (f/w)(x - y)| dy \leq \frac{|Q|}{w(Q)} \|K * \phi_r\|_1 \|f/w\|_\infty \leq c \|f/w\|_\infty.$$

Therefore,

$$\begin{aligned} \|K * (f/w)\|_{\text{BMO}_w} &\leq c \|K * (f/w)\|_{\text{BMO}}^* \\ &= \sup_{\substack{r > 0 \\ x \in \mathbb{R}^n}} \frac{|Q|}{w(Q)} \int |(K * \phi_r)(y) (f/w)(x - y)| dy \\ &\leq c \|f/w\|_\infty. \end{aligned}$$

Let H be a kernel such that $H^\varepsilon(x) = H(x) \cdot \chi_{\varepsilon < |x| < 1/\varepsilon}$ is integrable for each $\varepsilon > 0$. If $Sf(x) = \mathbf{p} \cdot \mathbf{v} \cdot H * f(x) = \lim_{\varepsilon \rightarrow 0} H^\varepsilon * f(x)$ exists, then by the dominated convergence theorem and (1.3), T is bounded from $L^\infty(\mathbb{R}^n)$ to $\text{BMO}(\mathbb{R}^n)$ if and only if

$$(1.4) \quad \sup_{\phi \in \Phi} \|S\phi_r\|_1 < \infty.$$

This lemma in the unweighted case is due to Peetre [6].

Proof of Theorem 1. Let b be a bounded function such that $H(x) = b(x)K(x)$ has a bounded Fourier transform. Set $Tf = p \cdot v \cdot H * f$. Suppose T is bounded from L^∞ to BMO . By (1.4) $\|T\phi_r\|_1 \leq c$ for all $r > 0$, $\phi \in \Phi$. In particular,

$$(1.5) \quad \|T\phi_r\|_{L^1(\mathbb{R}^n \setminus Q_{2\sqrt{n}r})} \leq c.$$

Let $K_\varepsilon(x) = K(x) \cdot \chi_{|x| > \varepsilon}$. We break $T\phi_r$ into two parts

$$\begin{aligned} T\phi_r(x) &= \lim_{\varepsilon \rightarrow 0} \int b(x-y)K_\varepsilon(x-y)\phi_r(y) dy \\ &= \lim_{\varepsilon \rightarrow 0} \int b(x-y)[K_\varepsilon(x-y) - K_\varepsilon(x)]\phi_r(y) dy + K(x)(b * \phi_r)(x) \\ &= g(x, r) + h(x, r). \end{aligned}$$

Note that if $x \in \mathbb{R}^n \setminus Q_{2\sqrt{n}r}$ and $y \in Q_r = Q_r(0)$, then $|x| \geq \sqrt{n}r \geq 2|y|$, hence $\mathbb{R}^n \setminus Q_{2\sqrt{n}r} \subseteq \{x \in \mathbb{R}^n : |x| \geq 2|y|\}$. Since K satisfies Hörmander's condition, (0.2) will be satisfied by all its truncates with a slightly larger c [8]. Now,

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus Q_{2\sqrt{n}r}} |g(x, r)| dx \\ &= \int_{\mathbb{R}^n \setminus Q_{2\sqrt{n}r}} \left| \lim_{\varepsilon \rightarrow 0} \int b(x-y)[K_\varepsilon(x-y) - K_\varepsilon(x)]\phi_r(y) dy \right| dx \\ &\leq \|b\|_\infty \sup_{\varepsilon > 0} \int \left(\int_{\mathbb{R}^n \setminus Q_{2\sqrt{n}r}} |K_\varepsilon(x-y) - K_\varepsilon(x)| dx \right) |\phi_r(y)| dy \\ &\leq \|b\|_\infty \sup_{\varepsilon > 0} \int \left(\int_{|x| \geq 2|y|} |K_\varepsilon(x-y) - K_\varepsilon(x)| dx \right) |\phi_r(y)| dy \\ &\leq c\|b\|_\infty \int |\phi_r(y)| dy \\ &= c \int |\phi(y)| dy = c, \end{aligned}$$

using Fubini's theorem along with the dominated convergence theorem. It follows from (1.5) that

$$\begin{aligned} \|K(x)(b * \phi_r)(x)\|_{L^1(\mathbb{R}^n \setminus Q_{2\sqrt{n}r})} &= \|h\|_{L^1(\mathbb{R}^n \setminus Q_{2\sqrt{n}r})} \\ &\leq \|g + h\|_{L^1(\mathbb{R}^n \setminus Q_{2\sqrt{n}r})} + \|g\|_{L^1(\mathbb{R}^n \setminus Q_{2\sqrt{n}r})} \leq c, \end{aligned}$$

hence (0.3) holds if we take supremum over all $\phi \in \Phi$ and $r > 0$. Conversely,

suppose (0.3) holds. Once again $\|T\phi_r\|_{L^1(R^n \setminus Q_{2\sqrt{nr}})} = \|g + h\|_{L^1(R^n \setminus Q_{2\sqrt{nr}})} \leq \|g\|_{L^1(R^n \setminus Q_{2\sqrt{nr}})} + \|h\|_{L^1(R^n \setminus Q_{2\sqrt{nr}})} \leq c$. Also $\|T\phi_r\|_{L^1(Q_{2\sqrt{nr}})} = \|T\phi_r \cdot \chi_{Q_{2\sqrt{nr}}}\|_1 \leq \|T\phi_r\|_{L^2(R^n)} \cdot \|\chi_{Q_{2\sqrt{nr}}}\|_{L^2(R^n)} = \|\hat{H} \cdot \hat{\phi}_r\|_{L^2(R^n)} \cdot cr^{n/2} \leq cr^{n/2} \sup_\xi |\hat{H}(\xi)| \cdot \|\hat{\phi}_r\|_{L^2(R^n)} = cr^{n/2} \|\phi_r\|_{L^2(R^n)} = cr^{n/2} \cdot r^{-n/2} = c$, using Schwarz's inequality and Plancherel's theorem. Thus $\|T\phi_r\|_1 = \|T\phi_r\|_{L^1(R^n \setminus Q_{2\sqrt{nr}})} + \|T\phi_r\|_{L^1(Q_{2\sqrt{nr}})} \leq c$. By (1.4), T is bounded. This completes the proof.

Next we show that there are bounded radial functions $b(|x|)$ for which (0.3) fails. Hence T , in general, does not map L^∞ into BMO. This is interesting in the light of the fact that if the kernel satisfies sufficiently strong regularity and cancellation properties, the L^2 bound is actually equivalent to the continuity of the mapping $L^\infty \rightarrow \text{BMO}$. See e.g. the T1 theorem of David and Journé [3]. Let

$$n = 2 \quad \text{and} \quad K(x_1, x_2) = \frac{x_1}{(x_1 + x_2)^{3/2}}$$

be the Riesz kernel. In polar coordinates it can be written as $K(r, \theta) = (\cos \theta / r^2) \cdot K$ is smooth away from zero. Define $b(|t|)$ and $\phi(x_1, x_2)$ as follows,

$$b(t) = \begin{cases} 1 & \text{if } 10m < |t| < 10m + 2 \\ -1 & \text{if } 10m - 2 < |t| < 10m \\ 0 & \text{otherwise} \end{cases}$$

where m runs through positive integers and

$$\phi(x_1, x_2) = \begin{cases} -1 & \text{if } 0 < x_1 < \frac{1}{2}, \quad -\frac{1}{2} < x_2 < \frac{1}{2} \\ 1 & \text{if } -\frac{1}{2} < x_1 < 0, \quad -\frac{1}{2} < x_2 < \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Claim. There are constants δ , η , and $c > 0$ such that $b * \phi(x) \geq c$ for $x \in \Gamma = \{x = re^{i\theta} : |\theta| < \delta, |r - 10m| < \eta \text{ and } m \geq 10\}$. Assuming the claim, it follows that

$$\begin{aligned} \int_{R^2 \setminus Q_{2\sqrt{2}}} |K(x)(b * \phi)(x)| dx &\geq \int_\Gamma |K(x)b * \phi(x)| dx \\ &\geq c \int_\Gamma |K(x)| dx \\ &\geq c \sum_{m=10}^{\infty} \int_{\substack{|r-10m|<\eta \\ |\theta|<\delta}} \frac{1}{|x|^2} dx \\ &= c \sum_{m=10}^{\infty} \int_{10m-\eta}^{10m+\eta} \frac{dr}{r} \\ &\geq c \sum_{m=10}^{\infty} \frac{1}{10m+\eta} = \infty. \end{aligned}$$

Let $\delta = 1/100$, $\eta = 1/10$. If $x^0 = (x_1^0, x_2^0) = r_0 e^{i\theta_0} \in \Gamma$, then $(b * \phi)(x^0) = \int_Q b(y)\phi(x^0 - y) dy$, where $Q = Q_1(x^0)$. By the particular choice of b , ϕ , δ and η it is easy to see that $\phi(x^0 - y)b(y) = 1$ on $D = \{(x_1, x_2) \in Q: |x_1 - x_1^0| > \frac{1}{5}\}$. Therefore,

$$\begin{aligned} (b * \phi)(x^0) &\geq \int_D b(y)\phi(x^0 - y) dy - \left| \int_{Q \setminus D} b(y)\phi(x^0 - y) dy \right| \\ &\geq |D| - |Q \setminus D| \\ &= \frac{3}{5} - \frac{2}{5} = \frac{1}{5}. \end{aligned}$$

Thus we can take $c = \frac{1}{5}$, and the claim is proved.

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