

# ON COMMON FIXED POINTS OF LINEAR CONTRACTIONS

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**ABSTRACT.** In this note we give a short and direct proof of a result about convex semigroups of linear contractions.

It is the aim of the present note to give a short proof of the following:

**Theorem.** *Let  $B$  be a Banach space such that  $B$  and its dual  $B^*$  are strictly convex. If  $S$  is a weakly compact convex semigroup of linear contractions on  $B$  then there exists a unique projection  $s_0 \in S$  such that  $s_0 s = s s_0 = s_0$  holds for all  $s \in S$ . Moreover,  $s_0 B$  is the set of common fixed points of the operators in  $S$ .*

Recall that a Banach space is called strictly convex if  $\|x\| = \|y\|$  and  $x \neq y$  imply  $\|(x + y)/2\| < \|x\|$ . The theorem above follows immediately from Corollary 4.14 and Theorem 7.2 in [1] but it appears that no direct proof of it has been published. A short proof in a special case was found by Radjavi and Rosenthal [3, Corollary 2]. The method of deLeeuw and Glicksberg is based on the theory of operator semigroups while in [3] Schauder's fixed point theorem is used. Our argument is based on the following simple fact: if  $K$  is a convex subset of  $B$  ( $B^*$ ) and  $K$  is compact in the weak (weak\*) topology then there exists a unique element of  $K$  with minimal norm.

For applications of the theorem we refer to [2, 4] (see also [1, §7] for connections with ergodic theory).

*Proof of the theorem.* Denote by  $B_0$  the set of common fixed points of the operators in  $S$ . Plainly  $0 \in B_0$  and  $B_0$  is a closed  $S$ -invariant subspace of  $B$ . Setting  $B_0^\perp := \{l \in B^* : l(x) = 0 \text{ for all } x \in B_0\}$  and  $S^* := \{s^* : s \in S\}$  we observe that  $B_0^\perp$  is a closed  $S^*$ -invariant subspace of  $B^*$  and that  $S^*$  is a weak\* compact convex semigroup of contractions on  $B^*$ . For every  $l \in B_0^\perp$  the set  $S^* l$  is convex and weak\* compact and hence there exists a unique  $l_0 \in S^* l$  with minimal norm. In view of  $s^* l_0 \in S^* l$  and  $\|s^* l_0\| \leq \|l_0\|$  we must have

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$s^*l_0 = l_0$  ( $s^* \in S^*$ ), that is

$$(1) \quad l_0(sx) = l_0(x), \quad s \in S, \quad x \in B.$$

We prove that  $l_0 = 0$ . Let  $x \in B$  be arbitrary and consider the weakly compact convex set  $Sx$ . By the same argument as above we see that there exists  $x_0 \in Sx$  having minimal norm and hence satisfying  $sx_0 = x_0$  ( $s \in S$ ). That is  $x_0 \in B_0$  and therefore  $l_0(x_0) = 0$ . Using (1) and the fact that  $x_0 = s'x$  with some  $s' \in S$  we obtain  $0 = l_0(x_0) = l_0(s'x) = l_0(x)$ . Thus  $l_0 = 0$ , i.e.,  $0 \in S^*l$  for every  $l \in B_0^\perp$ .

Let now  $l_1, \dots, l_n \in B_0^\perp$  be arbitrary and choose  $s_1^* \in S^*$  so that  $s_1^*l_1 = 0$ . Next we choose  $s_2^* \in S^*$  such that  $s_2^*(s_1^*l_2) = 0$ . Continuing this process we obtain an operator  $s^* = s_n^* \cdots s_2^*s_1^* \in S^*$  with  $s^*l_i = 0$  ( $i = 1, \dots, n$ ). A simple compactness argument shows the existence of an operator  $s_0^* \in S^*$  such that  $s_0^*l = 0$  for all  $l \in B_0^\perp$ , i.e.  $l(s_0x) = 0$  ( $l \in B_0^\perp, x \in B$ ). It follows that  $s_0B = B_0$ . Using the relation  $sx = x$  ( $x \in B_0, s \in S$ ) we obtain  $ss_0 = s_0$ .

It remains to prove that  $s_0s = s_0$ . The argument is suggested by that of Corollary 4.13 in [1]. Note first that  $s_0ss_0 = s_0$  and  $s_0ss_0s = s_0s$  hold because of  $ss_0 = s_0$ . For every  $l \in B^*$  we have

$$(2) \quad \|s^*s_0^*l\| = \|s^*s_0^*s^*s_0^*l\| \leq \|s_0^*s^*s_0^*l\| \leq \|s^*s_0^*l\|.$$

But  $s_0^*s^*s_0^* = s_0^*$  and therefore (2) gives  $\|s^*s_0^*l\| = \|s_0^*l\|$ . If  $s_0^*l \neq s^*s_0^*l$  for some  $l \in B^*$  then we would have

$$\|s_0^*l\| = \|s^*s_0^*l\| = \frac{1}{2}\|s_0^*(s^*s_0^*l + s_0^*l)\| \leq \frac{1}{2}\|s^*s_0^*l + s_0^*l\| < \|s_0^*l\|.$$

This contradiction shows that  $s_0^* = s^*s_0^*$  and therefore  $s_0s = s_0$ . The uniqueness of  $s_0$  follows at once from  $ss_0 = s_0s = s_0$ . The proof is complete.

The theorem has the following immediate corollary which generalizes Theorem 1 in [3].

**Corollary.** *Let  $B$  and  $S$  be as in the theorem. Then the operators in  $S$  have a common fixed point other than 0 if and only if the operator 0 is not in  $S$ .*

## REFERENCES

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