FINITE PROJECTIVE PLANES AND A QUESTION ABOUT PRIMES

WALTER FEIT

(Communicated by Andrew Odlyzko)

ABSTRACT. Let n be an even integer not divisible by 3. Suppose that $p = n^2 + n + 1$ is a prime and $2^{n+1} \equiv 1 \pmod{p}$. The question is asked whether this can only occur if n is a power of 2. It is noted that an affirmative answer to this question implies that a finite projective plane with a flag transitive collineation group is Desarguesian.

1. INTRODUCTION

Let n be an even natural number such that $p = n^2 + n + 1$ is a prime. The following additional hypotheses about n will be considered.

H1. $d^{n+1} \equiv 1 \pmod{p}$ for every divisor d of n .

H2. $n + 1 \equiv 0 \pmod{3}$ and $2^{n+1} \equiv 1 \pmod{p}$.

H3. $n \equiv 0 \pmod{8}$ and (H2) is satisfied.

If (H1) is satisfied then $n^{n+1} \equiv 1 \pmod{p}$. Thus $3 \mid (n + 1)$ since n has order 3 modulo p and so (H2) is satisfied.

If k is a natural number not divisible by 3 then a primitive cube root of unity is a root of $x^{2k} + x^k + 1$, thus $(x^2 + x + 1) \mid (x^{2k} + x^k + 1)$. Hence if $q = m^{2k} + m^k + 1$ is a prime for some integer $m > 1$ then k is a power of 3. In particular, if $m \neq 3$ is a prime then $m^{3k} \equiv 1 \pmod{q}$ and the order of m modulo q is a power of 3 and so is relatively prime to m . Hence $m^{m^k+1} \equiv 1 \pmod{q}$. Thus if $m = 2$ and $n = m^k$ then (H1) is satisfied. It is not known whether there are infinitely many primes of this form. Section 3 contains some numerical results in this connection. The following questions for $I = 1, 2, 3$ are related to the converse of this statement.

Question I. If (H1) is satisfied is n necessarily a power of 2?

If Question 2 has an affirmative answer then so do Questions 1 and 3. If Question 2 or 3 is to have an affirmative answer then the condition $n + 1 \equiv 0 \pmod{3}$ (which is equivalent to the condition $n \not\equiv 0 \pmod{3}$) is essential.

Received by the editors June 28, 1988, and, in revised form, February 17, 1989.

1980 *Mathematics Subject Classification.* (1985 Revision) Primary 05B10, 11A41, 51E15.

Key words and phrases. Finite projective plane, flag transitive, prime.

The work on this paper was partially supported by NSF grant DMS-8512904.

Otherwise for instance $n = 24$ provides a counter example. Here $p = 601$ and $2^{25} \equiv 1 \pmod{p}$.

I have verified that Questions 2 and 3 have an affirmative answer for $n \leq 14,400,008$ (and $p \leq 207,360,244,800,073$). This was done on a PC so that it should not be difficult to check the results for much larger values of n .

These questions may be of independent interest. In fact one could consider analogues of Question 1 without assuming that n is even. However in the form above they arise naturally in the study of finite projective planes.

A finite projective plane is said to be flag transitive if its collineation group is transitive on the set of flags (incident point-line pairs). It has been conjectured that a finite flag transitive projective plane is necessarily Desarguesian. The following result is relevant to this conjecture.

Theorem A. *Let π be a finite flag transitive projective plane of order n which is not Desarguesian. Then $p = n^2 + n + 1$ is a prime, $n \equiv 0 \pmod{8}$, n is not a power of 2 and (H1) is satisfied.*

Corollary. *If Question 3 has an affirmative answer then every finite flag transitive projective plane is Desarguesian.*

In view of the remarks above, Theorem A implies that a finite non-Desarguesian flag transitive projective plane has order $n > 14,400,008$.

The proof of Theorem A is quite short. However it is, amongst other things, based on a result of W. Kantor [4] which depends not only on the classification of the finite simple groups, but also on a detailed knowledge of all their maximal subgroups of odd index.

2. THE PROOF OF THEOREM A

Throughout this section π is a finite flag transitive non-Desarguesian projective plane, G is the collineation group of π . Thus there are $n + 1$ points on each line and $n^2 + n + 1$ points in π . By [4], Theorem A, n is even, $p = n^2 + n + 1$ is a prime and G is a Frobenius group of order $p(n + 1)$.

In particular, π is a cyclic plane. Thus if D is the subgroup of $F_p^\times$ of order $n + 1$ then D is a difference set modulo p . Furthermore, π and D determine each other up to isomorphism. See e.g. [3], Chapter 11.

Lemma 2.1. *(H1) and (H2) are satisfied.*

Proof. This follows from the multiplier theorem of M. Hall [3], Theorem 11.4.1. $\square$

Lemma 2.2. *If $n > 2$ then (H3) is satisfied.*

Proof. As (H2) is satisfied, 2 has odd order and so is a quadratic residue modulo p . Thus $p \equiv \pm 1 \pmod{8}$ and so $n \equiv 0$ or $2 \pmod{8}$. Hence it suffices to show that $n \not\equiv 2 \pmod{8}$. This result is known [5], [6], but we include a short proof here.

Let ζ be a primitive p^{th} root of unity and let $\alpha = \sum_{d \in D} \zeta^d$. Then $\alpha\bar{\alpha} = n$ as D is a difference set with $|D| = n + 1$. Let $K = \mathbf{Q}(\alpha) \subseteq \mathbf{Q}(\zeta)$. Then $[K : \mathbf{Q}] = n$ and α is the trace of ζ from $\mathbf{Q}(\zeta)$ to K . As $K \subseteq \mathbf{Q}(\zeta)$, 2 is not ramified in K . As $2^{n+1} \equiv 1 \pmod{p}$, 2 has residue class degree 1 in K . Therefore $(2) = \prod_{i=1}^n \mathfrak{p}_i$, where $\{\mathfrak{p}_i\}$ is a set of n distinct prime ideals of K . As $\alpha\bar{\alpha} = n = 2n_0$ with n_0 odd, it follows that exactly one of each pair $\mathfrak{p}_i, \bar{\mathfrak{p}}_i$ divides (α) . Therefore $\alpha + \bar{\alpha} \not\equiv 0 \pmod{\mathfrak{p}_i}$ for each i . As 2 has residue class degree 1, this implies that $\alpha + \bar{\alpha} + 1 \equiv 0 \pmod{\mathfrak{p}_i}$ for each i and so $\alpha + \bar{\alpha} + 1 \equiv 0 \pmod{2}$. The coefficient of 1 in $\alpha + \bar{\alpha} + 1$ is odd and $\alpha + \bar{\alpha} + 1$ is the sum of $2(n+1)+1$ p^{th} roots of unity. Any set of $p-1$ p^{th} roots of unity forms an integral basis in $\mathbf{Q}(\zeta)$. Therefore $2(n+1)+1 > p-1 = n(n+1)$. Hence $n \leq 2$ contrary to assumption. $\square$

Proof of Theorem A. In view of Lemmas 2.1 and 2.2 it suffices to show that n is not a power of 2. Suppose on the contrary that $n = 2^k$.

A result of Gordon, Mills and Welch, see [2] or [1], p. 89, implies that D consists of all the powers of 2 modulo p . Thus 2 has order $n+1$ modulo p . Since $2^{3k} \equiv n^3 \equiv 1 \pmod{p}$ this yields that $2^k + 1 = n + 1 \leq 3k$. Hence $k \leq 3$ and so $k = 1$ or 3. Thus $n = 2$ or 8. In these cases let D_0 be a difference set corresponding to the Desarguesian plane of order n . By the multiplier theorem D_0 can be chosen to consist of all the powers of 2 modulo p . Hence $D = D_0$ and so π is Desarguesian contrary to assumption. $\square$

3. SOME NUMERICAL RESULTS

For an integer $f \geq 0$ let $v(f) = n^2 + n + 1$, where $n = 2^k$ with $k = 3^f$. It is not known whether there are infinitely many values of f for which $v(f)$ is a prime.

TABLE 1

f	$d(f)$
3	2593
4	487
5	80191
6	39367
7	209953
8	141560137
9	472393
11	88219207
12	4361981634559
14	258280327
16	3357644239
17	12040253957143
19	662489036191
21	251048476873
23	35586121596607

If $0 \leq f \leq 2$ then $v(f)$ is a prime: $v(0) = 7, v(1) = 73$ and $v(2) = 262,657$. If $3 \leq f \leq 23$ then $v(f)$ is composite except possibly for $f = 10, 13, 15, 18, 20$ or 22 . Table I contains the smallest prime divisor $d(f)$ of $v(f)$ in the remaining cases. It may not be easy to check whether $v(f)$ is a prime for some of the values of f listed above since $v(13)$ is already larger than the largest known prime.

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DEPARTMENT OF MATHEMATICS, YALE UNIVERSITY, NEW HAVEN, CONNECTICUT 06520