PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 108, Number 3, March 1990

ANALOGUES OF TREYBIG'S PRODUCT THEOREM

H. MURAT TUNCALI

(Communicated by James E. West)

ABSTRACT. L. B. Treybig proved that a continuous Hausdorff image of a compact ordered space does not contain a nonmetric product of compact, infinite spaces. Analogous results hold for the images of rim-countable continua and locally connected, rim-scattered continua.

It is known, (see Treybig [3] and Bula [1]), that a compact ordered space cannot be mapped onto a nonmetric product of compact infinite spaces. In this paper, we will prove that analogous results hold for the classes of rim-countable continua and locally connected rim-scattered continua.

Throughout the paper all the spaces are assumed to be Hausdorff and all the mappings are assumed to be continuous. A continuum is a compact connected space. *A rim-countable continuum* is a continuum which admits a basis of open sets with countable boundaries. *A scattered set* in a topological space is a set which does not contain any nonempty, dense in itself subset; i.e., each nonempty closed subset has an isolated point. *A rim-scattered continuum* is a continuum which admits a basis of open sets whose boundaries are scattered.

We shall prove the following theorems:

Theorem 1. *Let X be a rim-countable continuum and $f\colon X \to Z$ a mapping of X onto a space Z. Then Z does not contain a product of a nonmetric, nondegenerate compact space and a perfect set.*

Theorem 2. *Let X be a rim-scattered, locally connected continuum and $f\colon X \to Z$ a mapping of X onto a space Z. Then Z does not contain a product of a nonmetric, nondegenerate compact space and a perfect set.*

Let X be a topological space. The *weight* $\mathrm{w}(X)$ of X is the least cardinal number α having the property that X admits a basis for its topology with cardinality $\leq \alpha$. The family N of subsets of X is said to be a *network* for X, if for each $x \in X$ and each open set $O \subset X$ containing x, there exists $V \in N$

Received by the editors October 15, 1987 and, in revised form, February 15, 1988. The results of this paper were presented at the Spring Topology Conference, Birmingham, Alabama, 1987.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54F25, 54F50; Secondary 54E35.

The contents of this paper constitute a part of the author's Ph.D. thesis under the supervision of Professor E. D. Tymchatyn.

such that $x \in V \subset O$. *The network weight* $\mathrm{nw}(X)$ of X is the least cardinal number α such that there exists a network with cardinality $\leq \alpha$. Note that for compact spaces $\mathrm{w}(X) = \mathrm{nw}(X)$ (Engelking [2, Theorem 3.1.19, p. 171]). We will denote the set of rational numbers by Q.

1. Rim-countable case

Lemma 3. *Let X be a rim-countable continuum and let U be an open F_σ-set in X. Then U has countably many componenets.*

Proof. Let U be an open F_σ-set in X. Then using the normality and the rim-countability of X, we can find a sequence of open sets V_n, $n = 1, 2, \ldots$, such that

$$V_1 \subset \mathrm{Cl}(V_1) \subset \cdots \subset V_n \subset \mathrm{Cl}(V_n) \subset \cdots \subset U$$

and $\mathrm{Bd}(V_n)$ is countable, and

$$U = \bigcup_{n=1}^{\infty} \mathrm{Cl}(V_n). \tag{I}$$

Let C be a componenet of $\mathrm{Cl}(V_n)$ for some n. Since $\mathrm{Cl}(V_n)$ is a closed subset of the continuum X, $C \cap \mathrm{Bd}(V_n) \neq \varnothing$, by the Boundary Bumping theorem. Thus, as $\mathrm{Bd}(V_n)$ is countable, $\mathrm{Cl}(V_n)$ has at most countably many components. From (I), it follows that U also has at most countably many components, because each component of $\mathrm{Cl}(V_n)$ lies in some component of U, and components are disjoint sets.

First we will prove the following theorem which is a special case of Theorem 1.

Theorem 4. *Let X be a rim-countable continuum and $f\colon X \to Z$ a mapping of X onto a space Z. Then Z does not contain a product of a nonmetric, nondegenerate compact space and* $[0, 1]$.

Proof. Let $Y \times [0, 1]$ be a subspace of Z such that Y is a nondegenerate compact space. We will show that Y is metrizable. Since Y is compact, as we noted $\mathrm{w}(X) = \mathrm{nw}(X)$. Therefore it suffices to show that Y has a countable network.

Let $\Pi_I\colon Y \times [0, 1] \to [0, 1]$ be the natural projection. Since $Y \times [0, 1]$ is a closed subset of the compact space Z, by the Tietze Extension Theorem, Π_I can be extended to an onto mapping $\Pi\colon Z \to [0, 1]$.

Let $r, s \in Q \cap [0, 1]$, $r < s$. Then $\Pi^{-1}((r, s))$ is an open set in Z containing $Y \times (r, s)$. Moreover $\Pi^{-1}((r, s))$ is an open F_σ-set. Indeed, we can find $r_n, s_n \in Q \cap [0, 1]$, $n = 1, 2, \ldots$, such that

$$r < \cdots < r_{n+1} < r_n < \cdots < r_1 < s_1 < \cdots < s_n < s_{n+1} < \cdots < s$$

and

$$\Pi^{-1}((r, s)) = \bigcup_{i=1}^{\infty} \Pi^{-1}((r_i, s_i)).$$

It follows that $f^{-1}(\Pi^{-1}((r,s)))$ is also an open F_σ-set in X and by Lemma 3, it has countably many components.

Let S_{rs} be the set of all components of $f^{-1}(\Pi^{-1}((r,s)))$. Let S be the union of all S_{rs}, $r,s \in Q \cap [0,1]$. Then S is a countable set, because it is the union of countably many countable sets. Let $N = \{\Pi_Y(f(C) \cap (Y \times [0,1])) : C \in S\}$ where $\Pi_Y : Y \times [0,1] \to Y$ is the natural projection onto Y. It is clear that N is a countable set. We shall show that N is a network for Y.

Let $y \in Y$ and let M be a closed subset of Y such that $y \in Y - M$. Then $\{y\} \times [0,1]$ and $M \times [0,1]$ are disjoint closed subsets in $Y \times [0,1]$, and hence in Z. Therefore, $f^{-1}(\{y\} \times [0,1])$ and $f^{-1}(M \times [0,1])$ are disjoint closed sets in X. Since X is rim-countable, there exists a closed countable set B such that B separates $f^{-1}(\{y\} \times [0,1])$ from $f^{-1}(M \times [0,1])$ in X. Since B is countable, there exists $t \in [0,1]$ such that

$$\text{(II)} \qquad f^{-1}(\Pi^{-1}(t)) \cap B = \varnothing.$$

Notice that $Y \times \{t\} \subset \Pi^{-1}(t)$. It follows that there exist $r,s \in Q \cap [0,1]$ such that $r < t < s$ and $f^{-1}(\Pi^{-1}((r,s))) \cap B = \varnothing$. Let C be a component of $f^{-1}(\Pi^{-1}((r,s)))$ such that for the point (y,t)

$$\text{(III)} \qquad C \cap f^{-1}((y,t)) \neq \varnothing.$$

By the definition of S, $C \in S$ and, therefore, $\Pi_Y(f(C) \cap (Y \times [0,1])) \in N$. By (II), $C \cap B = \varnothing$. So $C \cap f^{-1}(M \times [0,1]) = \varnothing$. This implies that $(f(C) \cap (Y \times [0,1])) \cap M \times [0,1] = \varnothing$. Hence, $\Pi_Y(f(C) \cap (Y \times [0,1])) \cap M = \varnothing$ and by (III), we have $y \in \Pi_Y(f(C) \cap (Y \times [0,1]))$. This proves that N is a network for Y.

Lemma 5. *Each compact perfect set maps onto* $[0,1]$.

Proof of Theorem 1. Assume that Z contains a product of a compact space Y and a perfect set F. By Lemma 5, there exists a mapping $g : F \to [0,1]$ of F onto $[0,1]$. In the proof of Theorem 4, replace the mapping Π_I with the mapping $g \circ \Pi_F$ (the composition of the mappings g and Π_F), where $\Pi_F : Y \times F \to F$ is the natural projection onto F, and follow the argument in the proof of Theorem 4.

Corollary 6. *A continuous image of a rim-countable continuum does not contain a nonmetric product of nondegenerate continua.*

2. Rim-scattered case

As in the previous case, we will first prove Theorem 2 for the case that the perfect set is $[0,1]$. To complete the proof, we will need the following lemmas, one of which is stated without proof.

Lemma 7. *Let X be a scattered compact space and let Y be a perfect space. Then there does not exist an onto mapping from X to Y.*

Proof. Suppose that there exists a mapping $f: X \to Y$ of X onto Y. Because of the compactness of X, we may assume that f is irreducible; i.e., no proper closed subset of X maps onto Y under f.

Let x be an isolated point of X. Then $\{x\}$ is an open set in X and, therefore, $X - \{x\}$ is closed. Since Y is perfect, $\{f(x)\}$ is not an isolated point of Y. Now, $f(X-\{x\})$ is a compact set, hence, closed subset of Y. Since $Y - \{f(x)\} \subset f(X - \{x\})$ and $f(x) \in \mathrm{Cl}(Y - f(x))$, we have $Y = f(X - \{x\})$ which contradicts the irreducibility of f.

Lemma 8. *Let X be a locally connected continuum and let U be an open F_σ-set in X. Then U has countably many components.*

Theorem 9. *If X is a rim-scattered, locally connected continuum and $f: X \to Z$ is a mapping of X onto Z, then Z does not contain the product of a nonmetric, nondegenerate compact space and* $[0, 1]$.

Proof. Suppose $Y \times [0, 1]$ is a subspace of Z where Y is a nondegenerate compact space. We will prove that Y is metrizable by showing that Y has a countable network as in the proof of Theorem 4.

Let $\Pi_I: Y \times [0, 1] \to [0, 1]$ be the natural projection. By the Tietze Extension Theorem, Π_I can be extended to a mapping $\Pi: Z \to [0, 1]$.

Let $r, s \in Q \cap [0, 1]$, $r < s$. Using Lemma 8, we construct the set S_{rs} as in the proof of Theorem 4. Let S be the union of all S_{rs}, $r, s \in Q \cap [0, 1]$ with $r < s$ and $N = \{\Pi_Y(f(C) \cap Y \times [0, 1])): C \in S\}$ where $\Pi_Y: Y \times [0, 1] \to Y$ is the natural projection. As in Theorem 4, N is a countable set.

The argument to show that N is a network for Y is the same as in the proof of Theorem 4 except that in this case the set B in the proof of Theorem 4 would be a scattered set. Applying Lemma 7 and following the same argument as in the proof of Theorem 4, we see that N is a network for Y.

Proof of Theorem 2. Theorem 9 and the arguments of the proof of Theorem 1 imply the result.

References

1. W. Bula, *Metrization theorems for continuous images of compact ordered spaces*, Topology and Ordered Structures, Part 2, M.C. Tracts 189, Math. Centrum, Amsterdam, 1983, pp. 23–29.
2. R. Engelking, *General topology*, PWN, Warsaw, 1977.
3. L. B. Treybig, *Concerning continuous images of compact ordered spaces*, Proc. Amer. Math. Soc. **15** (1964), 866–871.

Department of Mathematics, University of Saskatchewan, Saskatoon, Saskatchewan S7N 0W0 Canada