PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 108, Number 3, March 1990

ON UNITARY INVARIANT IDEALS IN THE ALGEBRA OF COMPACT OPERATORS

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(Communicated by John B. Conway)

ABSTRACT. Lie ideals are constructed, which are ideals in the algebra of compact operators but not ideals in the algebra of bounded operators, thus settling a question of C. K. Fong and H. Radjavi in the negative.

INTRODUCTION

Let $\mathscr{H}$ be an infinite dimensional complex Hilbert space. By operators we shall mean bounded linear transformations of $\mathscr{H}$ into itself. Their algebra is denoted by $\mathscr{B}(\mathscr{H})$, while $\mathscr{K}(\mathscr{H})$ is the algebra of all compact operators. By an ideal we mean a two-sided ideal. We say that $\mathscr{L} \subset \mathscr{B}(\mathscr{H})$ is a Lie ideal in $\mathscr{B}(\mathscr{H})$ if $\mathscr{L}$ is such a linear manifold, that $A \in \mathscr{B}(\mathscr{H})$ and $B \in \mathscr{L}$ imply $AB - BA \in \mathscr{L}$. Let $\mathscr{I}(\mathscr{A})$ and $\mathscr{J}(\mathscr{A})$ denote the ideals of $\mathscr{B}(\mathscr{H})$ and $\mathscr{K}(\mathscr{H})$, respectively, generated by the set $\mathscr{A} \subset \mathscr{K}(\mathscr{H})$. Clearly $\mathscr{I}(\mathscr{A}) \supseteq \mathscr{J}(\mathscr{A})$, but equality does not hold in general. For example if $a < 0$, $\mathscr{I}(\operatorname{diag}(n^a)) \neq \mathscr{J}(\operatorname{diag}(n^a))$ (see [2]). But in the case when $\mathscr{A}$ is countable and $\mathscr{J}(\mathscr{A})$ is a Lie ideal in $\mathscr{B}(\mathscr{H})$, we have $\mathscr{J}(\mathscr{A}) = \mathscr{I}(\mathscr{A})$, i.e. $\mathscr{J}(\mathscr{A})$ is an ideal in $\mathscr{B}(\mathscr{H})$ too (see [3, 4. Example]). The answer to whether $\mathscr{J}(\{A\}) = \mathscr{I}(\{A\})$ if $0 \leq A \in \mathscr{K}(\mathscr{H})$ appears in [2] along with the question: Is the above implication true in general, i.e. without any restriction on the cardinality of $\mathscr{A}$? In the present paper we discuss the case $\mathscr{A} = \mathscr{U}(A)$ where $0 \leq A \in \mathscr{K}(\mathscr{H})$ and $\mathscr{U}(A) = \{UAU^* : U^* = U^{-1}\}$, that is, when $\mathscr{A}$ is the unitary orbit of a positive compact operator. Then $\mathscr{J}(\mathscr{U}(A))$ is a unitary invariant manifold, so by [1, Theorem 1] and [3] it is also a Lie ideal in $\mathscr{B}(\mathscr{H})$ (we refer to [3] for generalization of any result in [1] to the nonseparable case). But it turns out that $\mathscr{J}(\mathscr{U}(A))$ is not an ideal in $\mathscr{B}(\mathscr{H})$ when e.g. $A = \operatorname{diag}(n^{-1})$.

Received by the editors November 3, 1988, and in revised form October 18, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 47D25, 47B10.

THE MAIN RESULT

For a compact operator A, $s_n(A)$ denotes the nth eigenvalue of the square-root of A^*A, $(n = 1, 2, \dots)$ (see [4]). We introduce the following notation:

$$S_n(A) = \sum_{i=1}^{N} s_i(A).$$

The following lemma is well-known (see e.g. [4, II. Lemma 4.1]):

Lemma 1. *If $A \in \mathscr{K}(\mathscr{H})$ and P is an arbitrary orthogonal projection with* $\operatorname{rank} P \leq n$*, then*

$$S_n(A) \geq |\operatorname{tr}(PAP)|.$$

Lemma 2. *If $0 \leq A \in \mathscr{K}(\mathscr{H})$ has infinite rank and either K or L is compact, then*

$$s_n(KAL) = o(s_n(A)).$$

If we also have $\sum s_n(A) = \infty$, then

$$S_n(KAL) = o(S_n(A)).$$

Proof. For the first statement we refer to [2, Lemma 1]. For the second one we note that $\{S_n(KAL)/S_n(A)\}_{n=1}^{\infty}$ is the transform of the zero-sequence $\{s_n(KAL)/s_n(A)\}_{n=1}^{\infty}$ with the regular matrix $T = [t_{ij}]$, where $t_{ij} = s_j(A)/S_i(A)$ if $j \leq i$ and zero otherwise.

Lemma 3. *If D is a positive operator and P and R are finite rank orthogonal projections with $P \leq R$, then*

$$\operatorname{tr}(RDR = \operatorname{tr}(PDP) + \operatorname{tr}((R-P)D(R-P)) \geq \operatorname{tr}(PDP).$$

Proof. The equality is obvious, if we choose an orthonormal basis of $\operatorname{ran} P$ and extend it to a basis of $\operatorname{ran} R$ and then to a basis of $\mathscr{H}$. For the inequality we note that $(R-P)D(R-P) \geq 0$.

Now we can prove the following

Theorem. *Suppose that $A \in \mathscr{K}(\mathscr{H})$, $0 \leq A$,*

$$\sum_{n=1}^{\infty} s_n(A) = \infty \tag{1}$$

and

$$\liminf_{n\to\infty} \frac{S_{kn}(A)}{S_n(A)} = 1 \tag{2}$$

for every positive integer k. Then $\mathscr{J}(\mathscr{U}(A)) \neq \mathscr{J}(\{A\})$.

Remarks. It is easy to prove that (2) with $k = 2$ implies (2) for every integer $k \geq 1$. Conditions (1) and (2) can be satisfied e.g. by $A = \operatorname{diag}(n^{-1})$ because then $S_n(A) = \log n + r_n$ for some $r_n \in [0, 1]$ and hence we can even write lim instead of lim inf in (2). But, for any nonincreasing sequence $\{\alpha_n\}$ with

$\alpha_n \to 0$ and $\sum \alpha_n = \infty$, we can select a subsequence $\{\alpha_{n_k}\}$, such that $\operatorname{diag}(\alpha_{n_k})$ satisfies the conditions of the theorem.

We know from [1, Theorem 1] that the unitary invariant linear submanifolds are exactly the Lie ideals in $\mathscr{B}(\mathscr{H})$. So $\mathscr{J}(\mathscr{U}(A))$ in the theorem is such a Lie ideal, which is an ideal in $\mathscr{K}(\mathscr{H})$ but is not an ideal in $\mathscr{B}(\mathscr{H})$ (see the question at the end of [2]). Particularly $\mathscr{J}(\mathscr{U}(A))$ is not countably generated as an ideal of $\mathscr{K}(\mathscr{H})$ (see [3]).

Proof. Let $s_n = s_n(A)$ and $S_n = S_n(A)$ for shortness. Let $\{e_n\}_1^\infty$ be an orthonormal sequence in $\mathscr{H}$ such that $Ae_n = s_n e_n$. Fix a strictly increasing sequence $\{n_k\}$ of positive integers, for which

$$(3) \qquad \lim_{k\to\infty} \frac{S_{N\cdot n_k}}{S_{n_k}} = 1$$

for every positive integer N. The existence of such a sequence follows from (2). By choosing a suitable subsequence if necessary, and calling it $\{n_k\}$, we can assume that

$$(4) \qquad \lim_{k\to\infty} \frac{S_{n_{k-1}}}{S_{n_k}} = 0$$

Clearly (3) remains true. Let $\mathscr{H}_k = \operatorname{span}(e_1, e_2, \dots, e_{n_k})$, and let P_k be the orthogonal projection of $\mathscr{H}$ onto $\mathscr{H}_k$. We need the following lemma:

Lemma 4. *Suppose that for the compact operator B we have*

$$(5) \qquad \liminf_{k\to\infty} \frac{S_{n_k}(B)}{S_{n_k}} = 0.$$

Suppose furthermore that there exist positive integers M, N, complex numbers a_i, unitary operators U_i and bounded operators K_j, L_j on $\mathscr{H}$ $(i = 1, 2, \dots, N,\ j = 1, 2, \dots, M)$ such that either K_j or L_j is compact $(j = 1, 2, \dots M)$ and

$$(6) \qquad 0 = B + \sum_{i=1}^{N} a_i U_i A U_i^* + \sum_{j=1}^{M} K_j A L_j.$$

Then $\sum_{i=1}^N a_i = 0$.

Proof. Let R_k be the orthogonal projection onto $\operatorname{span}(\bigcup_{i=1}^N U_i \mathscr{H}_k)$. Multiply both sides of (6) from the left and from the right by R_k, and take traces:

$$(7) \qquad 0 = \operatorname{tr}(R_k B R_k) + \sum_{i=1}^{N} a_i \operatorname{tr}(R_k U_i A U_i^* R_k) + \operatorname{tr}\left(\sum_{j=1}^{M} R_k K_j A L_j R_k\right).$$

Clearly $R_k \geq U_i P_k U_i^*$ $(k = 1, 2, \dots,\ i = 1, 2, \dots, N)$. So we have

$$\begin{aligned} S_{n_k} &= \operatorname{tr}(P_k A P_k) \leq \operatorname{tr}(U_i^* R_k U_i A U_i^* R_k U_i) \\ &= \operatorname{tr}(R_k U_i A U_i^* R_k) \leq S_{Nn_k}(U_i A U_i^*) = S_{Nn_k}. \end{aligned}$$

The first inequality follows from Lemma 3, since $U_i^* R_k U_i \geq P_k$, and the second one follows from Lemma 1, because $\operatorname{rank} R_k \leq N n_k$. Consequently we have

$$(8) \qquad \left|\operatorname{tr}(R_k U_i A U_i^* R_k) - S_{n_k}\right| \leq S_{Nn_k} - S_{n_k}.$$

Subtract $S_{n_k} \sum_1^N a_i$ from both sides of (7):

$$-\sum_{i=1}^N a_i S_{n_k} = \operatorname{tr}(R_k B R_k) + \sum_{i=1}^N a_i \left(\operatorname{tr}(R_k U_i A U_i^* R_k) - S_{n_k}\right) + \operatorname{tr}\left(\sum_{j=1}^M R_k K_j A L_j R_k\right).$$

Using (8) and Lemmas 1 and 2 we get

$$\begin{aligned}\left|-\sum_{i=1}^N a_i S_{n_k}\right| &\leq |\operatorname{tr}(R_k B R_k)| + (S_{Nn_k} - S_{n_k})\sum_{i=1}^N |a_i| + \sum_{j=1}^M |\operatorname{tr}(R_k K_j A L_j R_k)| \\ &\leq S_{Nn_k}(B) + (S_{Nn_k} - S_{n_k})\sum_{i=1}^N |a_i| + o(S_{Nn_k}) \qquad (k \to \infty).\end{aligned}$$

So

$$\left|\sum_{i=1}^N a_i\right| \leq \frac{S_{Nn_k}(B)}{S_{n_k}} + \left(\frac{S_{Nn_k}}{S_{n_k}} - 1\right)\sum_{i=1}^N |a_i| + \frac{o(S_{Nn_k})}{S_{Nn_k}}\frac{S_{Nn_k}}{S_{n_k}}.$$

This holds for every k. The third term converges to 0 as $k \to \infty$ and so does the second term by (3). Taking $\liminf$ of both sides we obtain

$$\left|\sum_{i=1}^N a_i\right| \leq \liminf_{k\to\infty} \frac{S_{Nn_k}(B)}{S_{n_k}} \leq N \liminf_{k\to\infty} \frac{S_{n_k}(B)}{S_{n_k}} = 0$$

by (5). This proves Lemma 4.

Now we return to the proof of the theorem. Let $F = \sum_{k=1}^\infty (P_{2k} - P_{2k-1})$, $P_0 = 0$, and $E = \sum_{k=1}^\infty (P_k - P_{k-1})$. First we put $B = FA$. We claim, that

$$\lim_{k\to\infty} \frac{S_{n_{2k+1}}(B)}{S_{n_{2k+1}}} = 0.$$

Indeed,

$$\begin{aligned} S_{n_{2k+1}}(B) &\leq S_{n_{2k+1}}((E - P_{2k+1} + P_{2k})A) \\ &= S_{n_{2k+1}+(n_{2k+1}-n_{2k})} - (S_{n_{2k+1}} - S_{n_{2k}}) \\ &\leq S_{2n_{2k+1}} + S_{n_{2k}} - S_{n_{2k+1}} \end{aligned}$$

which, by (3) and (4) proves our claim. So $B = FA$ satisfies condition (5) of Lemma 4. Similarly, the same can be proved for $B' = (E - F)A = A - FA$. Assume now, that (6) is satisfied by $B = FA$. Then

$$0 = A - FA + \sum_{i=1}^{N+1} (-a_i) U_i A U_i^* + \sum_{j=1}^M K_j A(-L_j)$$

where $a_{N+1} = 1$ and $U_{N+1} = id$. This means, that—although with different parameters— B' satisfies (6) too. So $\sum_{i=1}^{N} a_i = 0$ and $\sum_{i=1}^{N+1} -a_i = 0$, that is $a_{N+1} = 0$, a contradiction. Consequently (6) can hold for $B = FA$ with no parameters, in other words $FA \notin \mathscr{J}(\mathscr{U}(A))$. But clearly $FA \in \mathscr{J}(\mathscr{A})$, so the proof of the theorem is complete.

Traceable operators

Let $\mathscr{L}(A)$ be the Lie ideal in $\mathscr{B}(\mathscr{H})$ generated by $A \in \mathscr{B}(\mathscr{H})$. By [1, Theorem 1]

$$\mathscr{L}(A) = \operatorname{span}(\mathscr{U}(A)) =$$
$$\left\{ \sum_{i=1}^{N} a_i U_i A U_i^* : N \geq 0,\ a_i \in \mathbf{C},\ U_i^* = U_i^{-1}\ (i = 1, 2, \ldots N) \right\}.$$

Lemma 4 suggests the following

Definition. Let us say that an operator A is traceable if any of the following three equivalent conditions hold:

(i) There exists a linear functional f on $\mathscr{L}(A)$ such that $f(UAU^*) = 1$ for every unitary operator U.

(ii) $\sum_{i=1}^{N} a_i U_i A U_i^* = 0$ $(U_i^* = U_i^{-1})$ implies $\sum_{i=1}^{N} a_i = 0$.

(iii) $0 \notin \left\{ \sum_{i=1}^{N} \lambda_i U_i A U_i^* : \sum_{i=1}^{N} \lambda_i = 1 \right\}$.

For the implication (ii) $\Rightarrow$ (i) we (well-)define f by $f\left(\sum_{i=1}^{N} a_i U_i A U_i^*\right) = \sum_{i=1}^{N} a_i$. If A is traceable, we put $\mathscr{L}_0(A) = \ker f$. $\mathscr{L}_0(A)$ is a Lie ideal in $\mathscr{B}(\mathscr{H})$. If A is as in the theorem, then A is traceable even in the stronger sense that f can be extended to a unitary invariant linear functional $\hat{f}$ on $\mathscr{J}(\mathscr{U}(A))$ so that $\mathscr{J}(\mathscr{L}_0(A)) \subset \ker \hat{f}$. Assume furthermore that $A = A' \oplus 0 \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$. Then $A' \oplus (-A') \in \mathscr{L}_0(A)$, so $A \in \mathscr{J}(\mathscr{L}_0(A))$ but clearly $A \notin \mathscr{J}(\mathscr{L}_0(A))$, i.e. Lemma 4 itself gives a counterexample different from the first one.

It is clear that if the image of A in the Calkin algebra is a nonzero constant, then A is traceable. On the other hand, if the image of A is not a constant in the Calkin algebra, then A is not traceable. To see this, suppose that f is a unitary invariant linear functional on $\mathscr{L}(A)$. By a theorem of Topping (see [1, Corollary 1]) $\mathscr{L}(A) = \mathscr{B}(\mathscr{H}) = \mathscr{L}(\mathscr{P})$, where $\mathscr{P}$ is an orthogonal projection with $\dim\ker P = \dim\operatorname{ran} P$. But $P = P_1 + P_2$, where P_1 and P_2 are unitary equivalent to P. So $f(P) = 2f(P) = 0$, which implies $f \equiv 0$ on $\mathscr{B}(\mathscr{H}) = \mathscr{L}(A)$. This shows that A is not traceable. So,—assuming $\mathscr{H}$ is separable—among the noncompact operators we can identify the traceable ones. But in the case of compact operators the question seems more difficult. It is clear that if $K \in \mathscr{K}(\mathscr{H})$ then $K \oplus (-K)$ is not traceable, and that a trace-class operator with nonzero trace is traceable. The theorem of this paper gives an example of a traceable operator outside of trace-class. To see that even $0 \leq A \neq 0$ does not imply traceability, consider the following example:

$$\begin{aligned}
&2^{-1}\operatorname{diag}(1,0,0,2^{-1},2^{-1},0,0,4^{-1},4^{-1},4^{-1},4^{-1},0,0,0,0,0,0,0,0,\dots)\oplus 0\\
&\quad+2^{-1}\operatorname{diag}(2^{-1},1,0,0,0,2^{-1},0,0,0,0,0,4^{-1},4^{-1},4^{-1},4^{-1},0,0,0,0,\dots)\oplus 0\\
&\quad+2^{-1}\operatorname{diag}(2^{-1},0,1,0,0,0,2^{-1},0,0,0,0,0,0,0,0,4^{-1},4^{-1},4^{-1},4^{-1},\dots)\oplus 0\\
&\qquad=\operatorname{diag}(1,2^{-1},2^{-1},4^{-1},4^{-1},4^{-1},4^{-1},8^{-1},8^{-1},8^{-1},8^{-1},8^{-1},8^{-1},8^{-1},\dots)\oplus 0
\end{aligned}$$

It would be interesting to know what the traceable compact operators exactly are.

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