

TOPOLOGICAL REFLECTIONS REVISITED

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ABSTRACT. Two full reflective subcategories of \mathcal{Top} are constructed whose intersection is not reflective.

INTRODUCTION

Answering questions put by J. R. Isbell in 1964 and by H. Herrlich in 1967 (see [I] p. 33, [H₁]) we show that both the category of uniform spaces and the category of topological spaces contain two reflective subcategories* whose intersection is not reflective. This improves the result obtained in [AR] that large intersections of reflective subcategories of \mathcal{Top} need not be reflective.

In the first version of our paper we used deep topological constructions to get two reflective subcategories of \mathcal{Top} with a nonreflective intersection. It has turned out, however, that a much easier categorical approach yields the same result. We prove that every “reasonable” category has the following property: for each class \mathcal{M} of morphisms with a set of domains, the orthogonal subcategory \mathcal{M}^\perp is an intersection of two reflective subcategories. This solves the problem, since \mathcal{Top} is “reasonable” and in [AR] we have shown that it has a class \mathcal{M} of morphisms with a common domain such that \mathcal{M}^\perp is not reflective in \mathcal{Top} .

We also apply our technique to locally presentable categories. We have shown in [ART] that the set-theoretical Weak Vopěnka’s Principle (which states that \mathcal{Ord}^{op} cannot be fully embedded into \mathcal{Gra} , the category of graphs) is equivalent to the statement that intersections of reflective subcategories of locally presentable categories are always reflective. We now conclude the same result for finite intersections. More concretely: the category of graphs has two reflective subcategories with a nonreflective intersection iff Weak Vopěnka’s Principle is false.

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We work within Gödel–Bernays–von Neuman set theory, assuming the axiom of choice for classes. We are much indebted to R. Börger for pointing out several mistakes in an earlier version of our paper.

I. ORTHOGONAL SUBCATEGORIES

Recall that for each collection \mathcal{M} of morphisms of a category \mathcal{K} , \mathcal{M}^\perp denotes the subcategory of all objects K orthogonal to all \mathcal{M} -morphisms $m: A \rightarrow B$ (i.e. for each $g: A \rightarrow K$ there exists a unique $h: B \rightarrow K$ with $g = h \cdot m$).

Definition. A category is said to be *ranked* provided that each object K has a rank with respect to extremal monos, i.e. there exists a regular cardinal n such that $\text{hom}(K, -)$ preserves n -direct unions of extremal monos.

Remark. Ranked categories are precisely the bounded categories of P. J. Freyd and G. M. Kelly [FK] for the case of (epi, extremal mono)-factorizations. From Theorem 4.1.3 in [FK] it follows that every ranked, cocomplete and co-well-powered [and hence, (epi, extremal mono)-factorizable] category \mathcal{K} has the following property:

$$\Phi^\perp \cap \mathcal{L} \text{ is a reflective subcategory of } \mathcal{K}$$

for each small collection Φ of \mathcal{K} -morphisms and each full epireflective subcategory \mathcal{L} of \mathcal{K} . In fact, the conclusion of the mentioned theorem is that $(\Phi \cup \Psi)^\perp$ is reflective for each class Ψ of epimorphisms in \mathcal{K} . It is sufficient to apply the result to the class Ψ of all reflections of \mathcal{K} -objects. The only difficulty in applying the above theorem lies in the fact that in [FK] the basic category is always assumed to also be complete. It has been shown in [Ke] that this assumption is superfluous; e.g. from the proof of Theorem 10.2 in [Ke] it follows that the above theorem of [FK] holds without completeness (although the formulation of 10.2 is not sufficient).

Theorem. Let \mathcal{K} be a ranked, cocomplete, and co-well-powered category. Then for each class \mathcal{M} of morphisms all domains of which form a set the following holds: \mathcal{M}^\perp is an intersection of two reflective subcategories of \mathcal{K} .

Proof I. Denote by \mathcal{L} the subcategory of \mathcal{K} consisting of precisely those objects L such that for any $m: A \rightarrow B$ in \mathcal{M} and $g: A \rightarrow L$ there exists at most one $h: B \rightarrow L$ with $g = h \cdot m$. We will show that \mathcal{L} is epireflective in \mathcal{K} . The reflection of an object K of \mathcal{K} is performed stepwise by defining the following chain $K_i \xrightarrow{e_{ij}} K_j$ of epimorphisms $(i, j \in \text{Ord}, i \leq j)$:

(a) $K_0 = K$.

(b) Given K_i , stop if $K_i \in \mathcal{L}$, and else, find $m: A \rightarrow B$ in \mathcal{M} and distinct morphisms $h, h': B \rightarrow K$ with $h \cdot m = h' \cdot m$. Let $e_{i,i+1}: K_i \rightarrow K_{i+1}$ denote the coequalizer of h and h' .

(c) Given a limit ordinal i , define K_i and $(e_{ji})_{j < i}$ as a colimit of the preceding chain.

Since \mathcal{K} is co-well-powered, the construction eventually stops, yielding $K_i \in \mathcal{L}$. It is obvious that the epimorphism $c_{0i}: K \rightarrow K_i$ is a reflection of K in \mathcal{L} .

II. To prove the theorem, we present two reflective subcategories of \mathcal{L} (and hence, of \mathcal{K}) the intersection of which is \mathcal{M}^\perp . Observe first that the epi-reflectivity of \mathcal{L} clearly guarantees that $\mathcal{L} \cap \mathcal{K}^\perp$ is a reflective subcategory of \mathcal{L} for each small collection \mathcal{K} of \mathcal{K} -morphisms (by the remark above).

III. Denote by $\{A_t | t \in T\}$ the set of all domains of \mathcal{M} -morphisms. It is clearly possible to find nonempty sets $\mathcal{M}_{t,i} (t \in T \text{ and } i \in \text{Ord})$ of morphisms of \mathcal{K} such that all $\mathcal{M}_{t,i}$ -morphisms have the domain A_t and

$$\mathcal{M} = \bigcup_{t \in T} \bigcup_{i \in \text{Ord}} \mathcal{M}_{t,i}.$$

For each \mathcal{L} -object L and each morphism $g: A_t \rightarrow L$ denote by $d(g)$ the smallest ordinal i such that there exists $m: A_t \rightarrow B$ in $\mathcal{M}_{t,i}$ with $g \neq h \cdot m$ for all $h: B \rightarrow L$; put $d(g) = \infty$ if no such ordinal i exists (where we consider ∞ larger than all ordinals). Put

$$d(L) = \{(t, i) \in T \times \text{Ord} | d(g) = i \text{ for some } g: A_t \rightarrow L\},$$

and observe that $d(L)$ is a set since it can be coded by $\bigcup_{t \in T} \text{hom}(A_t, L)$ and T is a set. Observe that $(t, i) \in d(L)$ implies that i is an ordinal (not ∞) and $i > 0$. For each class $H \subseteq T \times \text{Ord}$ denote

$$\mathcal{L}_H = \{L \in \mathcal{L} | d(L) \cap H = \emptyset\}.$$

IV. For each set $H \subseteq T \times \text{Ord}$ we will prove that \mathcal{L}_H is a reflective subcategory of \mathcal{L} . By II, it is sufficient to find a set \mathcal{K} of \mathcal{K} -morphisms with $\mathcal{L}_H = \mathcal{L} \cap \mathcal{K}^\perp$. For each $(t, i) \in H$ let us form the multiple pushouts (in \mathcal{K}) of $\bigcup_{j < i} \mathcal{M}_{t,j}$ and of $\bigcup_{j \leq i} \mathcal{M}_{t,j}$:

$$A_t \xrightarrow{m} B_m \xrightarrow{\bar{m}} P_{t,i} \quad \left(m \in \bigcup_{j < i} \mathcal{M}_{t,j} \right)$$

and

$$A_t \xrightarrow{m'} B'_{m'} \xrightarrow{\bar{m}'} P'_{t,i} \quad \left(m' \in \bigcup_{j \leq i} \mathcal{M}_{t,j} \right).$$

We have the canonical morphism $c_{t,i}: P_{t,i} \rightarrow P'_{t,i}$ (defined by $c_{t,i} \cdot \bar{m} = \bar{m}'$ for all $m \in \bigcup_{j < i} \mathcal{M}_{t,j}$). We will verify that

$$\mathcal{L}_H = \mathcal{L} \bigcap \{c_{t,i} | (t, i) \in H\}^\perp.$$

In fact, let $L \in \mathcal{L}_H$. For each $f: P_{t,i} \rightarrow L$ put

$$g = f \cdot \bar{m} \cdot m: A_t \rightarrow L, \text{ independent of } m \in \bigcup_{j < i} \mathcal{M}_{t,j}.$$

(Observe that $i \neq 0$ and hence $\bigcup_{j < i} \mathcal{M}_{t,j} = \emptyset$.) Since g factors through each element of $\mathcal{M}_{t,j}$ for $j < i$, it follows that $d(g) \geq i$. Then $(t, i) \in H$ implies $d(g) \neq i$ (for else, $L \notin \mathcal{L}_H$), and hence, for each $m' \in \bigcup_{j \leq i} \mathcal{M}_{t,j}$ there exists $h_{m'}: B_{m'} \rightarrow L$ with $g = h_{m'} \cdot m'$. Consequently, there exists a unique $h: P'_{t,i} \rightarrow L$ with $h \cdot \bar{m}' = h_{m'}$ for all $m' \in \bigcup_{j \leq i} \mathcal{M}_{t,j}$. It follows that $f = h \cdot c_{t,i}$ (since $L \in \mathcal{L}$ implies $h_m = f \cdot \bar{m}$ for each m , and hence, $f \cdot \bar{m} = h \cdot c_{t,i} \cdot \bar{m}$) which proves $L \in \{c_{t,i}\}^\perp$.

Conversely, let $L \in \mathcal{L} \cap \{c_{t,i}|(t, i) \in H\}^\perp$. For each $g: A_t \rightarrow L$ with $d(g) = i \in \text{Ord}$ we are to show that $(t, i) \notin H$. Suppose the contrary, then for each $m \in \bigcup_{j < i} \mathcal{M}_{t,j}$ we have $h_m: B_m \rightarrow L$ with $g = h_m \cdot m$, and there exists a unique $f: P'_{t,i} \rightarrow L$ with $f \cdot \bar{m} = h_m$ ($m \in \bigcup_{j < i} \mathcal{M}_{t,j}$). Then $L \in \{c_{t,i}\}^\perp$ implies the existence of $h: P'_{t,i} \rightarrow L$ with $f = h \cdot c_{t,i}$ —thus, each $m \in \mathcal{M}_{t,i}$ fulfills $g = (h \cdot c_{t,i} \cdot \bar{m}) \cdot m$, in contradiction to $d(g) = i$.

V. We are going to find a disjoint decomposition

$$T \times \text{Ord} = H \cup \bar{H}$$

such that both \mathcal{L}_H and $\mathcal{L}_{\bar{H}}$ are reflective subcategories of \mathcal{L} . This will conclude the proof since

$$\mathcal{L}_H \cap \mathcal{L}_{\bar{H}} = \{L \in \mathcal{L} | d(g) = \infty \text{ for each } g: A_t \rightarrow L, t \in L\} = \mathcal{M}^\perp.$$

We first write the class \mathcal{L}^{ob} of all \mathcal{L} -objects in the form

$$\mathcal{L}^{ob} = \bigcup_{i \in \text{Ord}} \mathcal{L}_i$$

where each \mathcal{L}_i is small. Then we will define, by transfinite induction, sets H_i and \bar{H}_i ($i \in \text{Ord}$) such that

- (i) $H_i \cup \bar{H}_j \subseteq T \times \text{Ord}$ and $H_i \cap \bar{H}_j = \emptyset$ for all $i, j \in \text{Ord}$,
- (ii) for each class $H \subseteq T \times \text{Ord}$ such that $H_i \subseteq H \subseteq (T \times \text{Ord}) - \bar{H}_i$ all \mathcal{L}_i -objects have a reflection both in \mathcal{L}_H and in $\mathcal{L}_{\bar{H}}$, where $\bar{H} = (T \times \text{Ord}) - H$.

This will be sufficient because the classes $H = \bigcup_{i \in \text{Ord}} H_i$ and $\bar{H} = (T \times \text{Ord}) - H$ [$\supseteq \bigcup_{i \in \text{Ord}} \bar{H}_i$] then clearly satisfy the above requirement.

(a) First step. For each $t \in T$ choose an ordinal h_0 larger than any ordinal i with $(t, i) \in \bigcup_{L \in \mathcal{L}_0} d(L)$, and put

$$H_0 = \{(t, i) | t \in T, i < h_0\}.$$

Since H_0 is a set, each \mathcal{L} -object has a reflection $r_0: L \rightarrow L_0$ in \mathcal{L}_{H_0} , see IV. Choose an ordinal \bar{h}_0 larger than any ordinal i with $(t, i) \in \bigcup_{L \in \mathcal{L}_0} d(L_0)$, and put

$$\bar{H}_0 = \{(t, i) | t \in T, h_0 < i \leq \bar{h}_0\}.$$

It is our task to show that for each class $H \subseteq T \times \text{Ord}$ with $H_0 \subseteq H$ and $\bar{H}_0 \subseteq \bar{H} = (T \times \text{Ord}) - H$, all \mathcal{L}_0 -objects have a reflection in \mathcal{L}_H as well

as $\mathcal{L}_{\overline{H}}$. The latter is trivial since $\mathcal{L}_0 \subseteq \mathcal{L}_{\overline{H}}$: for each $L \in \mathcal{L}_0$ and each $(t, i) \in d(L)$ we have $i < h_0$ (by its choice of h_0) and thus $(t, i) \in H$ —consequently, $d(L) \cap \overline{H} = \emptyset$. Furthermore, each \mathcal{L}_0 -object L has a reflection in \mathcal{L}_{H_0} , viz., $r_0: L \rightarrow L_0$. In fact, $H_0 \subseteq H$ implies $\mathcal{L}_H \subseteq \mathcal{L}_{H_0}$, and since r_0 is a reflection in \mathcal{L}_{H_0} , it is sufficient to verify that $L_0 \in \mathcal{L}_H$. Each $(t, i) \in d(L_0)$ satisfies both $i \geq h_0$ (since $L_0 \in \mathcal{L}_{H_0}$) and $i < \overline{h}_0$ (by the choice of \overline{h}_0), and thus $(t, i) \in \overline{H}_0$. Consequently, $d(L_0) \cap H = \emptyset$.

(b) Induction step. Let i be an ordinal for which H_j and \overline{H}_j are already constructed for all $j < i$. We define H_i and \overline{H}_i as follows.

Since $\bigcup_{j < i} H_j$ is a set, each \mathcal{L} -object L has a reflection $r_i: L \rightarrow L_i$ in $\mathcal{L}_{\bigcup_{j < i} H_j}$, see IV. Choose an ordinal \overline{h}_i larger than each k with $(t, k) \in \bigcup_{L \in \mathcal{L}_i} d(L_i)$, and put

$$\overline{H}_i = \left\{ (t, k) \mid t \in T, k < \overline{h}_i \text{ and } k \notin \bigcup_{j < i} H_j \right\}.$$

We claim that whenever a class $H \subseteq T \times \text{Ord}$ fulfills $\bigcup_{j < i} H_j \subseteq H$ and $\overline{H}_i \subseteq \overline{H} = (T \times \text{Ord}) - H$, then each \mathcal{L}_i -object has a reflection in \mathcal{L}_H . In fact, $r_i: L \rightarrow L_i$ is such a reflection: since $\mathcal{L}_H \subseteq \mathcal{L}_{\bigcup_{j < i} H_j}$, it is sufficient to prove that $L_i \in \mathcal{L}_H$ (for each $L \in \mathcal{L}_i$). In fact, each $(t, k) \in d(L_i)$ satisfies both $k \notin \bigcup_{j < i} H_j$ (since $L_i \in \mathcal{L}_{\bigcup_{j < i} H_j}$) and $k < \overline{h}_i$ (by the choice of \overline{h}_i), and thus, $(t, k) \in \overline{H}_i$. Consequently, $d(L_i) \cap H = \emptyset$.

Analogously, using the (already established) \overline{H}_i , we know that each \mathcal{L} -object L has a reflection $\overline{r}_i: L \rightarrow \overline{L}_i$ in $\mathcal{L}_{\bigcup_{j \leq i} \overline{H}_j}$. Choose an ordinal h_i larger than each k with $(t, k) \in \bigcup_{L \in \mathcal{L}_i} d(\overline{L}_i)$, and put

$$H_i = \left\{ (t, k) \mid t \in T, k < h_i \text{ and } k \notin \bigcup_{j \leq i} \overline{H}_j \right\}.$$

Then each \mathcal{L}_i -object L has a reflection in $\mathcal{L}_{\overline{H}}$ whenever $H \subseteq T \times \text{Ord}$ fulfills $\bigcup_{j \leq i} H_j \subseteq H$ and $\bigcup_{j \leq i} \overline{H}_j \subseteq \overline{H} = (T \times \text{Ord}) - H$. In fact, $\overline{r}_i: L \rightarrow \overline{L}_i$ is such a reflection.

Problem. Is the hypothesis of a small collection of domains essential in the theorem? We can only present a category which has a subcategory which fails to be an intersection of a set (let alone of two!) reflective subcategories of \mathcal{K} . Nevertheless, \mathcal{K} is cocomplete, ranked, but not co-well-powered. (The example is a small adaptation of Example 5 in [RT].)

\mathcal{K} has objects $(X, P_i, R_i)_{i \in \text{Ord}}$ where X is a set, $P_i \subseteq X$ and $R_i \subseteq X \times X$ for $i \in \text{Ord}$, such that (1) $P_i \cap P_j = P_i \cap P_k$ for all $i \neq j$ and $i \neq k$; and (2) $R_i \cap R_j \neq \emptyset$ implies $R_i = R_k$ for all $i < j$ and $i < k$. Morphisms $f: (X, P_i, R_i) \rightarrow (X', P'_i, R'_i)$ are functions satisfying $f(P_i) \subseteq P'_i$ and $(f \times f)(R_i) \subseteq R'_i$.

($i \in \text{Ord}$). It is a routine verification to see that \mathcal{K} is a legitimate category which has all the above properties.

Consider the following morphisms $f_j: A_j \rightarrow B_j$ of \mathcal{K} ($j \in \text{Ord}$):

$$\begin{aligned} A_j &= (\{0\}, P_i, \emptyset), \\ P_j &= \{0\} \text{ and } P_i = \emptyset \quad \text{for all } i \neq j, \\ B_j &= (\{0, 1\}, P_i, R_i), \\ P_j &= \{0\}, \\ R_j &= \{(0, 1)\} \text{ and} \\ P_i &= R_i = \emptyset \quad \text{for all } i \neq j, f_j(0) = 0. \end{aligned}$$

The orthogonal subcategory $\{f_j\}^\perp$ consists of all \mathcal{K} -objects such that for each $x \in P_i$ there exists a unique y with $(x, y) \in R_i$. This subcategory is no intersection of a set of full, reflective subcategories of \mathcal{K} —the proof is analogous to that in [RT].

II. TOPOLOGICAL REFLECTIONS

We now turn to \mathcal{Top} and other concrete subcategories (where *concrete* means equipped with a faithful functor to *Set*). Recall that for concrete categories \mathcal{K} and \mathcal{L} an *almost full* embedding is an embedding $E: \mathcal{K} \rightarrow \mathcal{L}$ which either is full or (1) in \mathcal{L} each constant map carries a morphism and (2) for $X, Y \in \mathcal{K}$,

$$E(\text{hom}(X, Y)) = \text{hom}(EX, EY) - \{f|f: EX \rightarrow EY \text{ is a constant map}\}.$$

In [AR] we have proved the following result.

Proposition. *For the following category \mathcal{C} and for each concrete category \mathcal{K} with an almost full embedding $E: \mathcal{C} \rightarrow \mathcal{K}$, the subcategory $\{E\alpha_{0i}\}_{i \in \text{Ord}}^\perp$ is not reflective in \mathcal{K} . The objects of \mathcal{C} are A_i, B_i ($i \in \text{Ord}$) and C , the morphisms are freely generated by the following morphisms ($i, j, k \in \text{Ord}$):*

$$\begin{aligned} \alpha_{ij}: A_i &\rightarrow A_j && \text{for } i < j, \\ \beta_{ik}: A_i &\rightarrow B_k, \\ \gamma_i: C &\rightarrow A_i, \end{aligned}$$

and the following relations:

$$\begin{aligned} \alpha_{ij} &= \alpha_{ij} \cdot \alpha_{it} && \text{for } i < t < j, \\ \beta_{ik} &= \beta_{jk} \cdot \alpha_{ij} && \text{for } i < j \text{ (and all } k), \\ \beta_{ik} \cdot \gamma_i &= \beta_{kk} \cdot \gamma_k && \text{for } i > k. \end{aligned}$$

Corollary. *Let \mathcal{K} be a concrete, co-complete, co-well-powered, and ranked category with an almost full embedding of the above category \mathcal{C} into \mathcal{K} . Then \mathcal{K} has two reflective subcategories with a nonreflective intersection.*

Proof. In fact, the subcategory $\{E_{\alpha_0}\}^\perp$ is nonreflective, and by the above theorem, it is an intersection of two reflective subcategories.

Examples. (1) \mathcal{Top} (the category of topological spaces) has two reflective subcategories with a nonreflective intersection.

In fact, Koubek proved in [Ko] that the category $\mathcal{Top}_{3\frac{1}{2}}$ of all completely regular spaces has the following property: every concretizable category (in particular, \mathcal{C} above) has an almost full embedding into $\mathcal{Top}_{3\frac{1}{2}}$. Thus, \mathcal{Top} satisfies all hypotheses of the above corollary.

(2) \mathcal{Unif} (the category of uniform spaces) has two reflective subcategories with a nonreflective intersection.

To see this, use the same result of Koubek: since $\mathcal{Top}_{3\frac{1}{2}}$ has a full embedding into \mathcal{Unif} (via fine uniformities, see [I]), we have an almost full embedding of \mathcal{C} into \mathcal{Unif} . All other hypotheses of the above corollary are easy to verify; the fact that \mathcal{Unif} is ranked can be easily seen since extremal monos in \mathcal{Unif} are embeddings of subspaces, and each uniformly continuous pseudometric of a subspace can be uniformly extended to the whole space (see [I]).

(3) Whereas \mathcal{Top} and \mathcal{Unif} are certainly basic topological categories, the nonreflective intersections we have presented above are not quite illuminating. In contrast, in the category $2 - \mathcal{Top}$ of bitopological spaces we have a very illustrative example: the subcategory \mathcal{K}_i of all spaces whose i th topology is compact T_2 ($i = 1, 2$) is reflective in $2 - \mathcal{Top}$, but $\mathcal{K}_1 \cap \mathcal{K}_2$ is not (see [AR]).

Analogously, in the category of pseudotopological spaces there is an illustrative (but large) nonreflective intersection: the subcategory \mathcal{K} of all compact T_2 pseudotopological spaces (i.e. such that each ultrafilter has a unique limit) is not reflective (see [BK]). But the subcategory \mathcal{K}_α of all α -compact T_2 pseudotopological spaces (i.e. such that each ultrafilter with a member of cardinality $< \alpha$ has a unique limit) is reflective, and $\mathcal{K} = \bigcap_{\alpha \in \text{Card}} \mathcal{K}_\alpha$.

Remark. For locally presentable categories, the above theorem also clarifies the situation completely:

(1) We have proved in [ART] that in a set theory satisfying Weak Vopěnka's Principle, each subcategory of a locally presentable category \mathcal{K} closed under limits is already reflective, hence intersections of reflective subcategories are reflective. Weak Vopěnka's Principle states:

(WVP) \mathcal{Ord}^{op} cannot be fully embedded into \mathcal{Gra}

(\mathcal{Ord}^{op} is the dual to the ordered class of all ordinals, and \mathcal{Gra} is the category of graphs [=binary relations] and homomorphisms).

(2) The question of intersecting two reflective subcategories of a locally presentable category is, therefore, interesting only in a set theory satisfying the negation of Weak Vopěnka's Principle; briefly \neg WVP. Now, this assumption is certainly consistent with set theory because WVP implies the existence of measurable cardinals (and, conversely, the existence of huge cardinals implies that WVP is consistent), see [ART].

Proposition. Assuming \neg WVP, each of the following categories has two reflective subcategories with a nonreflective intersection:

$\mathcal{G}ra$,
 posets and strictly increasing maps,
 semigroups and homomorphisms,
 rings and homomorphisms.

Proof. Each of the mentioned categories is locally presentable and hence ranked, strongly cocomplete and co-well-powered. It remains to prove that \mathcal{C} above can be almost fully embedded in each of them. For $\mathcal{G}ra$ this has been (implicitly) performed in [ART]: we have a full embedding $E: \mathcal{C} \rightarrow \mathcal{G}ra$ where (in the notation of the proof of Theorem 3 of [ART]), $EC = FC_{3,5}$, $EA_i = \coprod_{j \leq i} D_j$, $EB_k = B_k$, $E\alpha_{ij}$ is the co-product injection, $E\beta_{ik} = \delta_{ik}$ and $E\gamma_i$ is the (unique) embedding of $FC_{3,5}$ into $D_i \rightarrow \coprod_{j \leq i} D_j$.

The rest is clear since, as proved in [PT], each of the remaining categories \mathcal{K} has an almost full embedding of $\mathcal{G}ra$ into \mathcal{K} .

REFERENCES

- [AR] J. Adámek and J. Rosický, *Intersections of reflective subcategories*, Proc. Amer. Math. Soc. **103** (1988), 710–712.
- [ART] J. Adámek, J. Rosický and V. Trnková, *Are all limit-closed subcategories of locally presentable categories reflective?* Proc. Categ. Conf. Louvain-La-Neuve 1987, Lecture Notes in Math. 1348, Springer-Verlag, 1988, 1–18.
- [BK] H.-P. Butzmann and G. Kneis, *Čech-Stone compactifications of pseudotopological spaces*, Math. Nachr. **128** (1986), 256–264.
- [FK] P. J. Freyd and G. M. Kelly, *Categories of continuous functors I*, J. Pure Appl. Algebra **2** (1972), 169–191.
- [H₁] H. Herrlich, *On the concept of reflections in general topology*, Proc. of Contributions to Extensions Theory of Topological Structures, Berlin, 1967.
- [H₂] —, *Topologische Reflexionen und Coreflexionen*, Lecture Notes in Math. 78, Springer-Verlag, Berlin, New York, Heidelberg 1968.
- [I] J. R. Isbell, *Uniform spaces*, Amer. Math. Soc., Providence, RI, 1964.
- [Ke] M. Kelly, *A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on*, Bull. Austral. Math. Soc. **22** (1980), 1–84.
- [Ko] V. Koubek, *Each concrete category has a representation by T_2 paracompact spaces*, Comment. Math. Univ. Carolin. **15** (1975), 655–664.
- [PT] A. Pultr and V. Trnková, *Combinatorial, algebraic and topological representations of groups, semigroups and categories*, North-Holland, Amsterdam, 1980.
- [RT] J. Rosický and W. Tholen, *Orthogonal and prereflective subcategories*, Cahiers Topologie Géom. Différentielle Catégoriques **29** (1988), 203–215.

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