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STARSHAPED UNIONS AND NONEMPTY INTERSECTIONS OF CONVEX SETS IN R^d

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ABSTRACT. Let $\mathcal{G}$ be a nonempty family of compact convex sets in R^d, $d \geq 1$. Then every subfamily of $\mathcal{G}$ consisting of $d+1$ or fewer sets has a starshaped union if and only if $\cap\{G : G \text{ in } \mathcal{G}\} \neq \emptyset$.

1. INTRODUCTION

We begin with some definitions. Let S be a subset of R^d. For points x and y in S, we say x *sees* y via S if and only if the corresponding segment $[x, y]$ lies in S. Set S is called *starshaped* if and only if there is some point p in S such that p sees via S each point of S, and the set of all such points p is the (convex) *kernel* of S.

A familiar theorem by Krasnosel'skii [4] states that for S a nonempty compact set in R^d, S is starshaped if and only if every $d+1$ points of S see via S a common point. In studying starshaped unions of sets, Kołodziejczyk [3] has proved that for $\mathcal{F}$ a finite family of closed sets in R^d, if every $d+1$ members of $\mathcal{F}$ have a starshaped union, then $\cup\{F : F \text{ in } \mathcal{F}\}$ is starshaped as well. In this paper, we examine the relationship between starshaped unions and nonempty intersections of compact convex sets in R^d to obtain the following Helly-type analogue: Let $\mathcal{G}$ be a nonempty family of compact convex sets in R^d, $d \geq 1$. Then every subfamily of $\mathcal{G}$ consisting of $d+1$ or fewer sets has a starshaped union if and only if $\cap\{G : G \text{ in } \mathcal{G}\} \neq \emptyset$. (Of course, when members of $\mathcal{G}$ have a nonempty intersection, they will have a starshaped union as well.) The proof is suggested by an argument of Klee [2].

Throughout the paper, $\operatorname{conv} S$, $\operatorname{int} S$, $\operatorname{bdry} S$, and $\ker S$ will denote the convex hull, interior, boundary, and kernel, respectively, for set S. For distinct points x and y, $L(x, y)$ will be the line they determine. The reader is referred

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to Valentine [6] and to Lay [5] for a discussion of related concepts and to Danzer, Grünbaum, Klee [1] for a survey of Helly-type results.

The results

The following definition is needed.

Definition. Set A is said to *surround* set B in the k-flat F, $k \geq 1$, if and only if A contains a $(k-1)$-sphere S such that B lies in the bounded component of $F \sim S$.

Our preliminary lemma is motivated by an argument of Klee [2].

Lemma 1. *Let $K_1, \ldots, K_l$ be nonempty compact convex sets in R^d, $d \geq 1$, $l \geq 2$, with $\cap\{K_i: 1 \leq i \leq l\} = \varnothing$ and with $a_i \in \cap\{K_j: 1 \leq j \leq l,\ j \neq i\} \neq \varnothing$ for $1 \leq i \leq l$. Then there are two flats H and L of dimension $l-1$ and $d-l+1$, respectively, meeting in a single point, such that*

(1) *$L \cap K_i = \varnothing$ and $a_i \in H$, $1 \leq i \leq l$, and*

(2) *$H \cap (\cup\{K_i: 1 \leq i \leq l\})$ surrounds $H \cap L$ in H.*

Proof. Clearly Helly's familiar theorem, together with the hypothesis of the lemma, imply that $2 \leq l \leq d+1$. We proceed by induction on d. If $d = 1$, then $l = 2$, and it is easy to see that the lemma holds. For $d > 1$, assume the result is true for integers k, $1 \leq k \leq d$, to prove for d. Since $\cap\{K_i: 1 \leq i \leq l\} = \varnothing$, let H_0 be a hyperplane strictly separating the compact convex sets K_1 and $\cap\{K_j: 2 \leq j \leq l\} \neq \varnothing$.

In case $l = 2$, let $H = L(a_1, a_2)$ and let $L = H_0$. If $l \geq 3$, choose $\{a_i'\} = [a_1, a_i] \cap H_0$, $2 \leq i \leq l$. Since $a_i' \in \cap\{K_j: j \neq 1, i\}$, every $l-2$ sets from $\{K_i \cap H_0: 2 \leq i \leq l\}$ have a nonempty intersection. However, H_0 is disjoint from $\cap\{K_j: 2 \leq j \leq l\}$, so $\cap\{K_i \cap H_0: 2 \leq i \leq l\} = \varnothing$. Using our induction hypothesis in the $(d-1)$-flat H_0, there exist flats H', L in H_0 having dimension $(l-1)-1 = l-2$ and $(d-1)-(l-1)+1 = d-l+1$, respectively, meeting in a single point, such that

(1) $L \cap K_i = \varnothing$ and $a_i' \in H'$, $2 \leq i \leq l$, and

(2) $H' \cap (\cup\{K_i: 2 \leq i \leq l\})$ surrounds $H' \cap L$ in H'.

Finally, let H be the flat determined by H' and a_1. Clearly $a_i \in L(a_1, a_i') \subseteq H$ for $2 \leq i \leq l$, and hence $a_i \in H$, $1 \leq i \leq l$. Moreover, since

$$\operatorname{bdry} \operatorname{conv}\{a_1, \ldots, a_l\} \subset H \cap (\cup\{K_i: 1 \leq i \leq l\}),$$

$H \cap (\cup\{K_i: 1 \leq i \leq l\})$ surrounds $H' \cap L = H \cap L$ in H. This finishes the induction and completes the proof of the lemma.

Theorem. *Let $\mathscr{G}$ be a nonempty family of compact convex sets in R^d, $d \geq 1$. Then every subfamily of $\mathscr{G}$ consisting of $d+1$ or fewer sets has a starshaped union if and only if $\cap\{G: G \text{ in } \mathscr{G}\} \neq \varnothing$.*

Proof. Clearly when $\cap\{G: G \text{ in } \mathscr{G}\} \neq \varnothing$, then every subfamily of $\mathscr{G}$ has a starshaped union whose kernel contains $\cap\{G: G \text{ in } \mathscr{G}\}$. Hence we need only establish the reverse implication.

Assume that every $d+1$ or fewer sets in $\mathcal{G}$ have a starshaped union, to show that $\cap\{G\colon G \text{ in } \mathcal{G}\} \neq \emptyset$. Note that for arbitrary sets G_1 and G_2 in $\mathcal{G}$, $G_1 \cup G_2$ is starshaped. Since both G_1 and G_2 are closed, this implies that $G_1 \cap G_2 \neq \emptyset$, and thus every two members of $\mathcal{G}$ intersect. By the familiar Helly theorem, it suffices to prove that every $d+1$ or fewer members of $\mathcal{G}$ have a nonempty intersection, $2 \leq d$.

Suppose on the contrary that for some maximal integer $l-1$, $2 \leq l-1 \leq d$, every $l-1$ members of $\mathcal{G}$ have a nonempty intersection but some l members of $\mathcal{G}$ have an empty intersection. Say $G_1 \cap \cdots \cap G_l = \emptyset$ for G_i in $\mathcal{G}$, $1 \leq i \leq l$. By Lemma 1, there exist flats H, L of dimension $l-1$, $d-l+1$, respectively, meeting in a single point, such that

(1) $L \cap G_i = \emptyset$, $1 \leq i \leq l$, and

(2) $H \cap (\cup\{G_i\colon 1 \leq i \leq l\})$ surrounds $H \cap L$ in H.

However, this contradicts the fact that $\cup\{G_i\colon 1 \leq i \leq l\}$ is starshaped. Our supposition is false, and $\cap\{G\colon G \text{ in } \mathcal{G}\} \neq \emptyset$, finishing the proof of the theorem.

Remark. It is interesting to observe that Theorem 1 holds without the requirement that members of $\mathcal{G}$ be compact, provided $\mathcal{G}$ is a finite family whose members are closed: In the proof, simply choose $x \in \ker(\cup\{G_i\colon 1 \leq i \leq l\}) \neq \emptyset$, $a_i \in \cap\{G_j\colon 1 \leq j \leq l\ j \neq i\}$, and define $T \equiv \operatorname{conv}\{x, a_i\colon 1 \leq i \leq l\}$. Then apply Lemma 1 to $\{T \cap G_i\colon 1 \leq i \leq l\}$. The finite version of Helly's theorem completes the argument.

However, the theorem fails without the restriction that members of $\mathcal{G}$ be closed, as the following easy example illustrates.

Example 1. Let $s_1, \ldots, s_{d+1}$ be vertices of a d-simplex in R^d, with $w \in \operatorname{int}\operatorname{conv}\{s_1, \ldots, s_{d+1}\}$. For $1 \leq i \leq d+1$, define

$$S_i = \operatorname{conv}\{w, s_j\colon 1 \leq j \leq d+1, j \neq i\}$$

and let $T_1 = S_1 \sim \{w\}$. Every d (or fewer) of the sets T_1, $S_2, \ldots, S_{d+1}$ intersect and hence have a starshaped union. Furthermore, $T_1 \cup S_2 \cup \cdots \cup S_{d+1} = \operatorname{conv}\{s_1, \ldots, s_{d+1}\}$ is convex and hence starshaped. However, $T_1 \cap S_2 \cap \cdots \cap S_{d+1} = \emptyset$.

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References

1. Ludwig Danzer, Branko Grünbaum, and Victor Klee, *Helly's theorem and its relatives*, Convexity, Proc. Sympos. Pure Math., Vol. 7, Amer. Math. Soc., Providence, RI, 1962, pp. 101–180.
2. V. L. Klee, *On certain intersection properties of convex sets*, Canad. J. Math. **3** (1951), 272–275.
3. Krzysztof Kołodziejczyk, *On starshapedness of the union of closed sets in* R^n, Colloq. Math. **53** (1987), 193–197.

4. M. A. Krasnosel'skii, *Sur un critère pour qu'un domaine soit étoilé*, Mat. Sb. (N.S.) (61) **19** (1946), 309–310.

5. Steven R. Lay, *Convex sets and their applications*, John Wiley, New York, 1982.

6. F. A. Valentine, *Convex sets*, McGraw-Hill, New York, 1964.

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