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A TRÜDINGER INEQUALITY ON SURFACES WITH CONICAL SINGULARITIES

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ABSTRACT. In this paper, the author establishes an isoperimetric inequality on surfaces with conical singularities, and by using it, proves a Trüdinger inequality with best constant on such surfaces. The best constants of the Trüdinger inequality are also found for a class of "symmetric" singular matrices.

0. INTRODUCTION

Recently Troyanov [1] systematically studied the surfaces with conical singularities, the singularities Picard considered. In order to prescribe Gaussian curvature on such surfaces, he proved the following Trüdinger inequality:

$$\int_S e^{bu^2}\,dA \le c_b \tag{0.1}$$

for all $u \in H^1(S)$ satisfying $\int_s |\nabla u|^2\,dA \le 1$ and $\int_S u\,dA = 0$ and for all $b < b_0$. Where S is a compact Riemannian surface with conical singularities of divisor $\beta = \sum \beta_i x_i$, dA is the area element of S, c_b is a constant related to b and $b_0 = 4\pi \min_i\{1, 1+\beta_i\}$. Although he claimed (0.1) to be true for $b = b_0$, it seems that he is unable to guarantee, by his method (Hölder inequality), that $\{c_b\}$ is bounded as $b \to b_0$ in the case $b_0 < 4\pi$. Hence, there are still some problems remaining unsolved:

(1) If (0.1) is true for $b = b_0 < 4\pi$?

(2) Is b_0 the best constant?

(3) (Posed by Troyanov) If the metric of the surface S is invariant under the action of some isometry group G, can the constant b in (0.1) be larger than b_0 for the class of G-equivariant functions? What is the best value of b in this case?

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In this paper, through an entirely different approach, we answer the above three questions completely. We prove the following:

Theorem I. *Let S be a compact Riemannian surface with conical singularities of divisor $\beta = \sum \beta_i x_i$, $b_0 = 4\pi \min_i\{1, 1+\beta_i\}$. Then the Trüdinger inequality*

$$(*) \qquad \int_S e^{b_0 u^2}\, dA \le c$$

holds for all $u \in H^1(S)$ satisfying $\int_S |\nabla u|^2\, dA \le 1$ and $\int_S u\, dA = 0$. Moreover b_0 is the largest possible constant, that is, if $b > b_0$, then there exists a sequence $\{u_k\} \subset H^1(S)$ with $\int_S |\nabla u_k|^2\, dA \le 1$ and $\int_S u_k\, dA = 0$, but $\int_S e^{bu_k^2}\, dA \to \infty$ as $k \to \infty$.

Theorem II. *Let G be some group of isometries on S, $G = \{g_1, g_2, \dots, g_s\}$. Let $I(x)$ be the number of distinct points in the set $\{g_1(x), g_2(x), \dots, g_s(x)\}$, $j_0 = \min_{x\in S} I(x)$ and $a_0 = 4\pi \quad \min_i\{I(x_i)(1+\beta_i), j_0\}$. Then*

$$(**) \qquad \int_S e^{a_0 u^2}\, dA \le c$$

holds for all $u \in H_G = \{v \in H^1(S) | v(g_k(x)) = v(x), \quad k = 1, \dots, s\}$ satisfying $\int_S |\nabla u|^2\, dA \le 1$ and $\int_S u\, dA = 0$. Moreover, a_0 is the best constant for $(**)$ to be true.

The proof of the theorems relies on an isoperimetric inequality on such surfaces.

In §1, based on the known isoperimetric inequalities on smooth surfaces, we proved a sort of isoperimetric inequality on the Riemannian surfaces with conical singularities.

In §2, using the isoperimetric inequality, thanks to the idea of Chang and Yang [2], we prove the inequality $(*)$. Then we find a counter example to show that b_0 is the best constant. And by our approach Theorem II follows easily.

1. An isoperimetric inequality on surfaces with conical singularities

In this section, in order to have some understanding of the surfaces with conical singularities, we first introduce two different local definitions of conical singularities, the relation between them and the global definition of such surfaces. Then we will prove a sort of isoperimetric inequality on these kinds of surfaces.

Let S be a surface, Σ a discrete subset of S and $S_0 = S \setminus \Sigma$. Assume that ds_0^2 is a Riemannian metric of class C^2 on S_0.

Definition 1.1 [1]. We say that $p \in \Sigma$ is a conical singular point of angle θ if there exist

(a) a neighborhood U of p and a number $\alpha > 0$,

(b) a diffeomorphism of class C^1: $h\colon (0,\alpha)\times \mathbb{R}/\theta\mathbb{Z} \to U_0 = U\setminus\{p\}$

(c) a continuous function $g\colon [0,\alpha)\times\mathbb{R}/\theta\mathbb{Z}\to\mathbb{R}$ satisfying the following conditions

(1) $h(r,t)\to p$ as $r\to 0$;

(2) for any t, the function: $r\to g(r,t)$ is of class C^2;

(3) $g(0,t)=0$ and $\partial g/\partial r(0,t)=1$ for all t;

(4) $(1/g)\partial^2 g/\partial r^2$ is continuous on $[0,\alpha)\times\mathbb{R}/\theta\mathbb{Z}$;

(5) $h^* ds_0^2 = dr^2 + g^2(r,t)\,dt^2$.

Definition 1.2 [1]. We say that $p\in\Sigma$ is a singularity of order α if there exist a neighborhood U of p, a homeomorphism $f\colon B=\{z\in\mathbb{C}|\,|z|<1\}\to U$ and a continuous function $u\colon B\to R$, such that

(1) $f(0)=p$, $f|_{B\setminus\{0\}}$ is a diffeomorphism

(2) $u|_{B\setminus\{0\}}$ is differentiable of class C^2

(3) $f^* ds_0^2 = e^{2u}|z|^{2\alpha}|dz|^2$.

Proposition 1.3 [1]. *Let S be a Riemannian surface of metric ds_0^2 with C^1 curvature. Then a point p on S is a singularity of order $\alpha>-1$ if and only if p is a conical singular point of angle $\theta=2\pi(\alpha+1)$.*

Now comes the global definition of the surfaces.

Definition 1.4 [1]. A Riemannian surface with conical singularity is a triple (S,β,ds^2) satisfying the following

(a) S is a compact Riemannian surface;

(b) β is a divisor >-1, i.e. a function: $S\to\mathbb{R}$ with discrete support;

(c) ds^2 is a conformal metric with bounded curvature and any point $p\in\operatorname{supp}\beta$ is a singularity of order $\alpha=\beta(p)$.

In the following we assume (S,β,ds_0^2) is a surface with conical singularities as defined above. Let $\operatorname{supp}\beta=\{x_1,x_2,\dots,x_m\}$ and $\beta(x_i)=\beta_i$. We also call S a surface with conical singularities of divisor $\beta=\Sigma\beta_i x_i$. Let $\beta_0=\min\{\beta_i\}$ and $\theta_0=2\pi\min\{1,1+\beta_0\}$.

Let γ be a simple closed curve separating S into two regions S_1 and S_2 with $A=\operatorname{area}(S_1)\le\operatorname{area}(S_2)$, and $L(\gamma)$ the length of γ. We are going to prove the following isoperimetric inequality:

Theorem 1.5. *There exist constants K, δ, $\varepsilon_0>0$, such that*

$$L^2(\gamma)/A \ge 2\theta_0 - KA, \qquad \text{for} A\le\delta \tag{1.1}$$

$$L^2(\gamma)/A \ge \varepsilon_0, \qquad \text{for all } A. \tag{1.2}$$

Proof.

Part I. In order to verify (1.1), we will separately consider the following three possibilities:

(a) $x_i\notin\overline{S}_1$, for all $i=1,2,\dots,m$;

(b) $x_i \in \overset{\circ}{S}_1$, the interior of S_1, for some i;
(c) $x_i \in \gamma$, for some i, but no $x_i \in \overset{\circ}{S}_1$.

Case (a). Let $N_\eta(S_1) = \{x \in S \mid \operatorname{dist}(x, S_1) < \eta\}$. Choose $\eta > 0$ so small that $x_i \notin N_\eta(S_1)$ for all $i = 1, 2, \ldots, m$. Now $(N_\eta(S_1), ds_0^2)$ is a smooth manifold with bounded curvature. By the known result of isoperimetric inequality on smooth surfaces (e.g. cf. [3], Theorem 6.), one can easily see that

$$L^2(\gamma)/A \geq 4\pi - KA \geq 2\theta_0 - KA, \qquad \text{for } A \text{ small}$$

where the constant K can be chosen as the upper bound of the curvature on $N_\eta(S_1)$.

Case (b). For sufficiently small A, we may assume that S_1 contains only one singularity, say x_1 of order β_1. By the definition of (S, β, ds_0^2), there is a neighborhood U of x_1, a homeomorphism $f\colon B = \{z \in \mathbb{C} \mid |z| < 1\} \to U$ and a continuous function $u\colon B \to R$, such that

(1) $f(0) = x_1$, $f|_{B\setminus\{0\}}$ is a diffeomorphism;
(2) $u|_{B\setminus\{0\}}$ is differentiable of class C^2;
(3) $f^* ds_0^2 = e^{2u}|z|^{2\beta_1}|dz|^2$.

Let $\theta_1 = 2\pi \min\{1, 1 + \beta_1\}$, we want to show that

$$L^2(\gamma)/A \geq 2\theta_1 - KA, \qquad \text{for } A \text{ small.} \tag{1.3}$$

We may assume that $L^2(\gamma)/A \leq 2\theta_1$, otherwise we are done. And under this assumption we can let A be so small that $S_1 \subset U$. Write $\widetilde{D} = f^{-1}(S_1)$. Then

$$A = \int_{\widetilde{D}} e^{2u}|z|^{2\beta_1} \frac{i}{2}\, dz \wedge d\overline{z}$$

and

$$L(\gamma) = \int_{\partial\widetilde{D}} e^{u}|z|^{\beta_1}|dz| \qquad \text{where } i = \sqrt{-1}.$$

First we consider the case $-1 < \beta_1 < 0$. Define a mapping $w = w(z)$ from z-plane to w-plane by

$$w(z) = \frac{2\pi}{\theta_1} z^{\theta_1/2\pi}.$$

Then by a straightforward computation, it is easy to see that $|z|^{2\beta_1}|dz|^2 = |dw|^2$, and $|z|^{2\beta_1}\, dz \wedge d\overline{z} = dw \wedge d\overline{w}$.

Let $D = w(\widetilde{D})$, then

$$A = \int_D e^{2u(z(w))}(i/2)\, dw \wedge d\overline{w}$$

and

$$L(\gamma) = \int_{w(\partial\widetilde{D})} e^{u(z(w))}|dw|.$$

Let $w = re^{i\theta}$. Without loss of generality, we may assume that $w(\partial\widetilde{D})$ in w-plane can be characterized by

$$r = f(\theta), \qquad 0 \le \theta \le \theta_1$$

and write $v(r,\theta) = u(z(w))$, then

$$A = \int_0^{\theta_1}\int_0^{f(\theta)} e^{2v(r,\theta)} r\,dr\,d\theta \tag{1.4}$$

and

$$L(\gamma) = \int_0^{\theta_1} e^{v(f(\theta),\theta)}\left[(f'(\theta))^2 + f^2(\theta)\right]^{1/2} d\theta. \tag{1.5}$$

In order to compare A with $L(\gamma)$, we try to exploit some known isoperimetric inequality on w-plane with some smooth metric. To this end, we first relate $w(\partial\widetilde{D})$ to a closed curve on w-plane by the equation

$$\tilde{f}(\theta) = f(\theta\theta_1/2\pi) \qquad \text{for } 0 \le \theta \le 2\pi.$$

It is obvious that the curve $\tilde{\gamma}$ defined by

$$r = \tilde{f}(\theta), \qquad 0 \le \theta \le 2\pi$$

is a simple closed curve on w-plane.

Represent $w(\partial B)$ by the equation: $r = g(\theta)$, $0 \le \theta \le \theta_1$. Let $\tilde{g}(\theta) = g(\theta\theta_1/2\pi)$, $0 \le \theta \le 2\pi$. Denote G the region on w-plane enclosed by the curve: $r = \tilde{g}(\theta)$, $0 \le \theta \le 2\pi$. Obviously, $\tilde{\gamma}$ is a closed curve in G. Let the metric on G be defined by

$$d\tilde{s}^2 = e^{2v(r,\theta\theta_1/2\pi)}|dw|^2, \qquad \text{with } w = re^{i\theta}.$$

Then on the manifold $(G, d\tilde{s}^2)$ the length of $\tilde{\gamma}$ is

$$L(\tilde{\gamma}) = \int_0^{2\pi} e^{v(\tilde{f}(\theta),\theta\theta_1/2\pi)}\left[(\tilde{f}'(\theta))^2 + \tilde{f}^2(\theta)\right]^{1/2} d\theta \tag{1.6}$$

and the area enclosed by $\tilde{\gamma}$ is

$$\widetilde{A} = \int_0^{2\pi}\int_0^{\tilde{f}(\theta)} e^{2v(r,\theta\theta_1,2\pi)} r\,dr\,d\theta. \tag{1.7}$$

We will prove that the curvature $\widetilde{K}(w)$ of $(G, d\tilde{s}^2)$ is bounded. Now let us assume this fact for a moment, and set $\widetilde{K} = \sup_G \widetilde{K}(w)$. Then by a known result concerning isoperimetric inequalities (e.g. cf. [3] or [4]) for smooth surfaces, we have

$$L^2(\tilde{\gamma})/\widetilde{A} \ge 4\pi - \widetilde{K}\widetilde{A}. \tag{1.8}$$

Taking into account (1.4)–(1.7), by a change of variables and straightforward calculation, one easily finds that

$$(2\pi/\theta_1)L(\gamma) \ge L(\tilde{\gamma}) \qquad \text{and} \qquad \widetilde{A} = (2\pi/\theta_1)A.$$

And this, together with (1.8), lead to our desired inequality

$$L^2(\gamma) \geq 2\theta_1 A - \widetilde{K} A^2.$$

Now in order to complete the proof of (1.3), what is left to be done is to show that the curvature of the manifold $(G, d\tilde{s}^2)$ is bounded.

In fact, let $K_0(x)$ be the curvature of (S, ds_0^2), then by Definition 1.4,

$$K_0(x) \text{ is bounded.} \tag{1.9}$$

In the neighborhood U of x_1, we can write (cf. [1])

$$K_0(f(z)) = -\Delta_z u e^{-2u} |z|^{-2\beta_1} \tag{1.10}$$

where $\Delta_z = 4\partial^2/\partial z \partial \overline{z}$.

In the domain $w(B)$, the image of the unit ball B on z-plane, consider the metric

$$ds^2 = e^{2u(z(w))} |dw|^2$$

with the Gaussian curvature $K(w)$. Then by the formula

$$K(w) = -\Delta_w u(z(w)) e^{-2u(z(w))}$$

as well as (1.10), one can easily derive that

$$K(w) = K_0(f(z)).$$

While by the definition of ds^2 and $d\tilde{s}^2$, it is easily seen that there is a constant c, such that

$$\widetilde{K}(w) \leq c K(w). \tag{1.11}$$

Now (1.9) and (1.11) imply the existence of a constant K, such that

$$\widetilde{K}(w) \leq K \qquad \text{for all } w \in G.$$

Moreover by (1.9), such a K can be chosen to be independent of the neighborhood U we consider.

Similarly, one can show that the inequality (1.3) holds for $\beta_1 \geq 0$.

Finally, note that $\theta_1 \geq \theta_0$, we have completed the proof of (1.1) for case (b).

Case (c). Again let A be so small that there is only one singular point, say x_1, belongs to the boundary of S_1 and no singular point is in the interior of S_1. Let the domain $\widetilde{D}$ on z-plane and the mapping w from z-plane to w-plane be the same as in Case (b). Then it is obvious that $w(\partial \widetilde{D})$ is a closed curve on w-plane. By the known isoperimetric inequality on smooth surfaces we have

$$L^2(\gamma)/A \geq 4\pi - KA.$$

This implies (1.1).

Part 2. Inequality (1.2) is the easy consequence of (1.1) and the following two obvious facts

(1) The area of S is finite.

(2) If there is a sequence $\{\gamma_k\}$ with $L(\gamma_k) \to 0$ as $k \to \infty$, then A_k, the smaller area enclosed by γ_k, goes to 0 as $k \to \infty$.

In fact, suppose that (1.2) were false. Then there would exist a sequence $\{\gamma_k\}$ and the corresponding $\{A_k\}$, such that

$$L^2(\gamma_k)/A_k \to 0, \qquad \text{as } k \to \infty.$$

But $A_k \le (\frac{1}{2})\,\text{area}(S)$, one must have $L(\gamma_k) \to 0$ as $k \to \infty$, and this implies $A_k \to 0$. Now by inequality (1.1), we see that for k sufficiently large,

$$L^2(\gamma_k)/A_k \ge \theta_0$$

a contradiction. This completes the proof of Theorem 1.5.

Remark 1.6. One may prove a stronger version of Theorem 1.5. However, to our goal of proving the Trüdinger inequality, the theorem is enough.

2. The proof of Theorems I and II

Let $H^1(S)$ be the Hilbert space of functions u that satisfy

$$\int_S (|\nabla u|^2 + u^2)\, dA < +\infty$$

(1) The proof of the Trüdinger inequality

$$(*) \qquad \int_S e^{b_0 u^2}\, dA \le C$$

is similar to that of Chang and Yang [2] with minor modifications. However, for completeness, we would rather sketch the whole proof here.

The following well-known calculus inequality plays an important role in the proof.

Lemma 2.1 (Moser) [5]. *Suppose $w(t)$ is a monotonically increasing function defined on the real line $(-\infty, +\infty)$ satisfying*

$$(2.1) \qquad \int_{-\infty}^{+\infty} \dot{w}^2(t)\, dt \le 1 \text{ and } \int_{-\infty}^{+\infty} w(t)\rho(t)\, dt = 0$$

with $\rho(t)$ a positive continuous function satisfying

$$(2.2) \qquad \rho(t) \le c_0 e^{-|t|} \quad \text{and} \quad \int_{-\infty}^{+\infty} \rho(t)\, dt = 1$$

for some constant c_0. Then

$$(2.3) \qquad \int_{-\infty}^{+\infty} e^{w^2(t)} \rho(t)\, dt \text{ is uniformly bounded.}$$

The proof of inequality $(*)$ is divided into two steps. Step I deals with C^2 functions u defined on S which have only isolated nondegenerate critical points, i.e. Morse functions. This will be done by change of variables based on the distribution function of u and using Lemma 2.1 above. In Step II we will use an approximation argument to show $(*)$ for all functions in $H^1(S)$.

Step I. Let u be a C^2 Morse function defined on S. For each real number M, let

$$L_M = \text{ length of the level curve } \{u = M\}$$
$$A_M = \text{ area of the region } \{u \le M\}.$$

It follows that

$$\int_S |\nabla u|^2 \, dA \ge \int_{-\infty}^{+\infty} \frac{L_M^2}{dA_M/dM} dM \qquad \text{and} \int_S u \, dA = \int_{-\infty}^{+\infty} M \, dA_M.$$

Then the assumption in Theorem I becomes

$$\int_{-\infty}^{+\infty} \frac{L_M^2}{dA_M/dM} dM \le 1 \qquad \text{and} \qquad \int_{-\infty}^{+\infty} M \, dA_M = 0 \tag{2.4}$$

and the inequality we are going to prove becomes

$$\int_{-\infty}^{+\infty} e^{bM^2} \, dA_M \le C. \tag{2.5}$$

Now again let γ be a simple closed curve on S enclosing area A, $L(\gamma)$ the length of γ and $|S|$ the area of the surface (S, β, ds_0^2). Define

$$I(A) = \begin{cases} \inf\limits_{\gamma}(L^2(\gamma)/A) & \text{if } 0 < A < |S|/2 \\ \inf\limits_{\gamma}\{L^2(\gamma)/(|S| - A)\} & \text{if } |S|/2 \le A < |S| \end{cases}$$

where inf is taken among all such curves which enclose a region with area A. Then by Theorem 1.5, we have $I(A) \ge \varepsilon_0$ for all A and

$$I(A) \ge 2\theta_0 - KA \qquad \text{for } A \text{ small}$$
$$I(A) \ge 2\theta_0 - K(|S| - A) \qquad \text{for } |S| - A \text{ small.}$$

Choose a C^1 function $\varphi(A)$ such that

$$I(A)/|S| \ge \varphi(A) \ge \varepsilon_0/|S| \qquad \text{for all } A,$$

and

$$\varphi(A) = \begin{cases} (2\theta_0 - KA)/|S| & \text{for } A \text{ small} \\ [2\theta_0 - K(|S| - A)]/|S| & \text{for } |S| - A \text{ small.} \end{cases}$$

Obviously,

$$L_M^2 \ge \varphi(A_M) A_M (|S| - A_M) \qquad \text{for any } A_M \in (0, |S|). \tag{2.6}$$

In order to apply Lemma 2.1, we now make a change of variable by defining $t \in (-\infty, \infty)$ as a function of M:

$$t = b \int_{|S|/2}^{A_M} [\varphi(A_M) A_M(|S| - A_M)]^{-1} \, dA_M$$

and $w(t) = \sqrt{b} \cdot M$, $\rho(t) = (dA_M/dt)(1/|S|)$. Then (2.4) and (2.6) imply (2.1) and it is obvious that $\int_{-\infty}^{+\infty} \rho(t)\, dt = 1$. The verification that such a t is well defined and $\rho(t) \le c_0 e^{-|t|}$ can be found in article [2]. Now by Lemma 2.1, for $b \le b_0$, we arrive at (2.3), then (2.5), and then the inequality $(*)$. This completes Step I.

Step II. Note that for any $u \in H^1(S)$, there exists a sequence of C^2 functions $\{u_k\}$, $u_k \to u$ in $H^1(S)$, it is easily seen that $(*)$ holds for all $u \in H^1(S)$. (Cf. [2] or [6] Theorem 2.47.)

(2) The proof that b_0 is the best constant in the inequality $(*)$ is a special case $(G = \{\mathrm{id}\})$ of the proof of the second part of Theorem II.

Proof of Theorem II. From the proof of Theorem I, it can be seen that in order to prove the Trüdinger inequality $(**)$ for the constant a_0, one needs only to show that for functions $u \in H_G$,

$$L_M^2 / A_M \ge a_0 - K A_M, \qquad \text{as } A_M \text{ small.} \tag{2.7}$$

In fact, let D_M be the region $\{u \le M\}$. Then for small A_M we may assume that each connected component of A_M contains either only one singular point or no singular point. It can be seen easily that there is an open set D in D_M such that $D_M = \bigcup_{i=1}^{s} g_i(D)$ and either the elements in $\{g_1, (D), \dots, g_s(D)\}$ are disjoint or identical. Assume that there are k disjoint sets in $\{g_1(D), \dots, g_s(D)\}$ denoted by $D_1, D_2, \dots, D_k$. Then $D_M = \bigcup_{i=1}^{k} D_i$. Let γ_i and A_i be the boundaries and areas of D_i, respectively. Then

$$L(\gamma_1) = \cdots = L(\gamma_k), \qquad \text{and} \qquad A_1 = \cdots = A_k.$$

By Theorem 1.5, one can easily infer that $L^2(\gamma_i)/A_i \ge b_0 - K A_i$ with

$$b_0 = \begin{cases} 4\pi, & \text{if } D \text{ contains no singular point} \\ 4\pi \min\{1 + \beta_i\}, & \text{if } D \text{ contains singular points } x_1, \dots, x_{i_0}, \\ & i = 1, \dots, i_0. \end{cases}$$

Consequently,

$$L_M^2 / A_M = k L_1^2 / A_1 \ge k b_0 - k K A_1 = k b_0 - K A_M. \tag{2.8}$$

(i) If $b_0 \ge 4\pi$, then since $k \ge j_0$, we have

$$k b_0 = 4\pi j_0 \ge a_0 = 4\pi \min\{I(x_i)(1 + \beta_i), j_0\}.$$

(ii) If $b_0 = 4\pi(1 + \beta_i)$ for some i, then since $k \ge I(x_i)$ we also have

$$k b_0 \ge 4\pi I(x_i)(1 + \beta_i) \ge a_0.$$

Therefore (2.8) implies (2.7) and we have proved the first part of the theorem.

In order to show that a_0 is the best constant in $(**)$, we are going to construct a sequence of functions $\{u_k\} \subset H_G$, such that

$$\int_S |\nabla u_k|^2 \, dA \le 1, \qquad \text{and} \qquad \int_S u_k \, dA = 0$$

but

$$\int_S e^{bu_k^2} \, dA \to +\infty \qquad \text{as } k \to \infty$$

for any $b > a_0$.

(1) Assume $I(x_1)(1+\beta_1) = \min\{I(x_1)(1+\beta_i), j_0\}$, i.e. $a_0 = 4\pi I(x_1)(1+\beta_1)$. Let $\theta_0 = 2\pi(1+\beta_1)$. For simplicity write $I = I(x_1)$ and let $p_1, p_2, \dots, p_I$ be the I distinct points in the set $\{g_1(x_1), \dots, g_s(x_1)\}$.

By [1] (Also cf. Definition 1.1.), there exist a neighborhood U of p_1, a number $\alpha > 0$ and a diffeomorphism

$$h\colon (0,\alpha) \times \mathbb{R}/\theta_0\mathbb{Z} \to U_0 = U \setminus \{p_1\}$$

such that

$$h^* \, ds_0^2 = dr^2 + g^2(r,t)\, dt^2$$

where g is a continuous function: $[0,\alpha) \times \mathbb{R}/\theta_0\mathbb{Z} \to \mathbb{R}$ satisfying (1)–(5) in Definition 1.1. As a consequence

$$\frac{1 - g(r,t)/r}{r^2} \quad \text{is bounded.} \tag{2.9}$$

Let $0 < \lambda < \rho < \alpha$, define

$$u_\lambda(r) = \begin{cases} \sqrt{ln(\rho/\lambda)/\theta_0} & \text{for } 0 \le r \le \lambda \\ \frac{ln(\rho/r)}{\sqrt{\theta_0 ln(\rho/\lambda)}} & \text{for } \lambda < r \le \rho \\ 0 & \text{for } \rho < r. \end{cases}$$

Let $|x - p|$ be the distance from point x to p on S. Then by (2.9),

$$\begin{aligned} &\int_S |\nabla u_\lambda(|x-p_1|)|^2 \, dA \\ &= \int_0^{\theta_0} \int_\lambda^\rho \frac{1}{\theta_0 ln(\rho/\lambda) r^2} g(r,t)\, dr\, dt \\ &\le \frac{1}{ln(\rho/\lambda)} \int_\lambda^\rho \tfrac{1}{r}(1 + cr^2)\, dr \\ &= 1 + c(\rho^2 - \lambda^2)/2\ln(\rho/\lambda) \to 1 \qquad \text{as } \lambda \to 0. \end{aligned} \tag{2.10}$$

Also by (2.9) and an elementary calculation, we have

$$\begin{aligned} \int_S u_\lambda(|x-p_1|)\, dA &\le \int_0^{\theta_0} \int_0^\rho u_\lambda(r) g(r,t)\, dr\, dt \\ &\le c\theta_0 \int_0^\rho u_\lambda(r) r\, dr \to 0 \qquad \text{as } \lambda \to 0. \end{aligned} \tag{2.11}$$

In order to show that

$$\int_S e^{b(u_\lambda(|x-p_1|))^2}\,dA \to +\infty \qquad \text{as } \lambda \to 0, \quad \text{for any } b > 4\pi(1+\beta_1) \tag{2.12}$$

we apply (2.9) again to get $g(r, t)/r \to 1$ as $r \to 0$. Hence, there exists $\lambda_0 > 0$, such that for $r \le \lambda_0$

$$g(r, t) \ge r/2.$$

Then for any $\lambda \le \lambda_0$

$$\begin{aligned}\int_S e^{b(u_\lambda(|x-p_1|))^2}\,dA &\ge \theta_0/2 \int_0^\lambda (\rho/\lambda)^{b/\theta_0} r\,dr \\ &= (\theta_0/4)\rho^{b/\theta_0}\lambda^{2-b/\theta_0} \to +\infty \qquad \text{as } \lambda \to 0\end{aligned} \tag{2.13}$$

because $b/\theta_0 > 2$ for $b > 4\pi(1+\beta_1)$.

Now let $v_\lambda(x) = \sum_{i=1}^{I} u_\lambda(|x-p_i|)$,

$$\tilde{v}_\lambda(x) = \frac{v_\lambda(x) - \frac{1}{|S|}\int_S v_\lambda(x)\,dA}{\left(\int_S |\nabla v_\lambda|^2\,dA\right)^{1/2}}.$$

It is easy to verify that $\tilde{v}_\lambda(x) \in H_G$, i.e. for any isometry transformation g on S, $\tilde{v}_\lambda(gx) = v_\lambda(x)$. By (2.10) and (2.11) we have

$$\tilde{v}_\lambda = I^{-1/2} v_\lambda + o(1)$$

where $o(1) \to 0$ as $\lambda \to 0$.

For any $a > a_0$, let a_1 be a constant that $a > a_1 > a_0$, then for λ sufficiently small

$$\int_S e^{a\tilde{v}_\lambda^2}\,dA \ge \int_S e^{(a_1/I)\tilde{v}_\lambda^2}\,dA \ge \int_S e^{(a_1/I)[u_\lambda(|x-p_1|)]^2}\,dA.$$

Since $a_1/I > a_0/I = 4\pi(1+\beta_1)$, by (2.13) we know that the right-hand side of the above inequality goes to $+\infty$. Therefore

$$\int_S e^{a\tilde{v}_\lambda^2}\,dA \to +\infty \qquad \text{as } \lambda \to 0.$$

(2) For the case $j_0 < \min_i\{I(x_i)(1+\beta_i)\}$, choose a point x_0 on S, such that $\{g(x_1), \dots, g(x_0)\}$ has exactly j_0 distinct points. If we take x_0 instead of x_1 in the proof of Case (1), then the proof follows similarly.

This completes the proof of Theorem II.

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