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CONVERGENCE OF DIFFERENCE APPROXIMATIONS AND NONLINEAR SEMIGROUPS

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ABSTRACT. We give a convergence theorem for the difference approximation for the evolution equation $(d/dt)u(t) \in Au(t)$ and a generation theorem of nonlinear semigroups for "directed" dissipative operators A in a real Hilbert space.

INTRODUCTION

In this paper we consider the evolution equation

$$\text{(DE)} \qquad (d/dt)u(t) \in Au(t), \qquad t \in [0, T)$$

for a given (multivalued) nonlinear operator A in a real Hilbert space.

After a pioneering work of Kōmura [4], many authors have treated the generation of nonlinear semigroups having dissipative or ω-dissipative operator as generators. Our main purpose is to introduce a notion of directed L-dissipative operators as a generalization of ω-dissipative operators and to give a convergence theorem for difference approximations for (DE) and a generation theorem of nonlinear semigroups through it. In §1 we state definitions and a convergence theorem for difference approximation for (DE) and the proofs are given in §2. In §3 we give a generation theorem of nonlinear semigroups through the convergence theorem in §1.

1. DEFINITIONS AND CONVERGENCE OF DIFFERENCE APPROXIMATIONS

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let A be a (multivalued) operator in H with domain $D(A)$ and range $R(A)$. We say that A is directed L-dissipative if the following conditions are satisfied:

(i) for every $\lambda > 0$, $(1 - \lambda A)^{-1}$ is a single valued operator.
(ii) $\langle x' - y', x - y \rangle \leq L(\|x - y\|)\|x - y\|$ for x, $y \in D(A)$, $x' \in Ax$ and $y' \in Ay$, where L is an increasing continuous function such that $L(0)$

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equals zero and satisfies the condition

$$\int_0^1 dt/L(t) = \infty.$$

(iii) $\langle x' - x'', \lambda x' \rangle \le 0$ for $\lambda > 0$, $x \in D(A)$, $x' \in Ax$, $x - \lambda x' \in D(A)$ and $x'' \in A(x - \lambda x')$.

We note that the condition (iii) is an analogue to that of "temporal analyticity" in Furuya [2]. Our main theorem is the following:

Theorem 1. *Let A be a directed L-dissipative operator in H satisfying the range condition*

$$\text{(R1)} \qquad R(1 - \lambda A) \supset D(A) \qquad \text{for all } \lambda > 0.$$

Let $T > 0$ and $T = nh_n$, where n is a positive integer. Let $u_0 \in D(A)$. We define the simple functions u_n on $[0, T]$ by

$$u_n(t) = (1 - h_n A)^{-k} u_0 \qquad \text{for } t \in ((k-1)h_n, kh_n]$$

and $u_n(0) = u_0$. Then $\{u_n(t)\}$ converges uniformly on $[0, T]$ as $n \to \infty$ and if we let u be the limit function of $\{u_n\}$, then u is a Lipschitz continuous function with Lipschitz constant $|Au_0| \equiv \inf\{\|z\| \mid z \in Au_0\}$, i.e.

$$(1) \qquad \|u(t) - u(s)\| \le |Au_0||t - s| \qquad \text{for } t, s \in [0, T].$$

2. Proof of Theorem 1

We begin with the following lemma:

Lemma 2.1. *Let A be a directed L-dissipative operator in H satisfying the range condition* (R1). *Let x_0, $y_0 \in D(A)$. Let h, k be positive numbers such that $h = \ell k$, where ℓ is a positive integer. Put $x_{n+1} = (1 - hA)^{-1} x_n$ and $y_{m+1} = (1 - kA)^{-1} y_m$ for n, $m = 0, 1, 2, \ldots$. Let $M = \max(|Ax_0| + |Ay_0|, |Ay_0|^2)$, where $|Ax| \equiv \inf\{\|x'\| \mid x' \in Ax\}$. Let ε be a given positive number. If $\|x_n - y_{\ell n}\| \ge \varepsilon$ and $Mh < \min(\varepsilon/2, L(\varepsilon)\varepsilon)$, then the following estimate holds*:

$$(2) \qquad \|x_{n+1} - y_{\ell(n+1)}\| - \|x_n - y_{\ell n}\| \le 3hL(\|x_n - y_{\ell n}\| + Mh).$$

Proof. Let $x'_{n+1} \in Ax_{n+1}$. Let p be a nonnegative integer. Choose $y'_{\ell(n+1)-p} \in Ay_{\ell(n+1)-p}$ such that $y_{\ell(n+1)-p} = y_{\ell(n+1)-p-1} + ky'_{\ell(n+1)-p}$. Then

$$\begin{aligned}
(3) \quad & \|x_{n+1} - pkx'_{n+1} - y_{\ell(n+1)-p}\|^2 \\
&= \|x_{n+1} - (p+1)kx'_{n+1} - y_{\ell(n+1)-p-1} + k(x'_{n+1} - y'_{\ell(n+1)-p})\|^2 \\
&= \|x_{n+1} - (p+1)kx'_{n+1} - y_{\ell(n+1)-p-1}\|^2 \\
&\quad + 2k\langle x'_{n+1} - y'_{\ell(n+1)-p}, x_{n+1} - y_{\ell(n+1)-p}\rangle \\
&\quad - (p+1)k^2\|x'_{n+1} - y'_{\ell(n+1)-p}\|^2 - pk^2\|x'_{n+1}\|^2 \\
&\quad + pk^2\|y'_{\ell(n+1)-p}\|^2 \qquad \text{for } p = 0, 1, 2, \ldots, \ell - 1.
\end{aligned}$$

Choose $x'_{n+1} \in Ax_{n+1}$ such that $x_n = x_{n+1} - hx'_{n+1}$. Then, for any $x'_n \in Ax_n = A(x_{n+1} - hx'_{n+1})$,

$$\begin{aligned} 0 &\geq \langle x'_{n+1} - x'_n, x'_{n+1}\rangle = \|x'_{n+1}\|^2 - \langle x'_n, x'_{n+1}\rangle \\ &\geq \|x'_{n+1}\|^2 - \|x'_n\|\|x'_{n+1}\|. \end{aligned}$$

Thus the following estimates hold:

$$\|x'_{n+1}\| \leq \|x'_n\| \leq |Ax_0|. \tag{4}$$

$$\|x_{n+1} - x_n\| = \|hx'_{n+1}\| \leq h|Ax_0|. \tag{5}$$

$$\begin{aligned} &|\|x_{n+1} - y_{\ell(n+1)-p}\| - \|x_n - y_{\ell_n}\|| \\ &\quad \leq \|x_{n+1} - x_n\| + \sum_{j=1}^{\ell-p} \|y_{\ell n+j} - y_{\ell n+j-1}\| \\ &\quad \leq h|Ax_0| + (\ell - p)k|Ay_0| \\ &\quad \leq Mh \qquad \text{for } p = 0, 1, 2, \ldots, \ell. \end{aligned} \tag{6}$$

From (3), using (4), (5), (6) and (ii) we have

$$\begin{aligned} &\|x_{n+1} - pkx'_{n+1} - y_{\ell(n+1)-p}\|^2 - \|x_{n+1} - (p+1)kx'_{n+1} - y_{\ell(n+1)-p-1}\|^2 \\ &\quad \leq 2kL(\|x_n - y_{\ell n}\| + Mh)(\|x_n - y_{\ell n}\| + Mh) + Mpk^2. \end{aligned}$$

Adding these inequalities for $p = 0, 1, 2, \ldots, \ell - 1$, we get

$$\begin{aligned} &\|x_{n+1} - y_{\ell(n+1)}\|^2 - \|x_n - y_{\ell n}\|^2 \\ &\quad \leq 2hL(\|x_n - y_{\ell n}\| + Mh)(\|x_n - y_{\ell n}\| + Mh) + M\ell(\ell - 1)k^2/2 \\ &\quad \leq 2hL(\|x_n - y_{\ell n}\| + Mh)(\|x_n - y_{\ell n}\| + Mh) + Mh^2. \end{aligned}$$

In addition, if $\|x_n - y_{\ell n}\| \geq \varepsilon > 2Mh$, then using (6) we have

$$\begin{aligned} &\|x_{n+1} - y_{\ell(n+1)}\|^2 - \|x_n - y_{\ell n}\|^2 \\ &\quad = (\|x_{n+1} - y_{\ell(n+1)}\| - \|x_n - y_{\ell n}\|)(\|x_{n+1} - y_{\ell(n+1)}\| + \|x_n - y_{\ell n}\|) \\ &\quad \geq (\|x_{n+1} - y_{\ell(n+1)}\| - \|x_n - y_{\ell n}\|)(2\|x_n - y_{\ell n}\| - Mh) \\ &\quad \geq (\|x_{n+1} - y_{\ell(n+1)}\| - \|x_n - y_{\ell n}\|)(\|x_n - y_{\ell n}\| + Mh). \end{aligned}$$

Thus, if $\|x_n - y_{\ell n}\| \geq \varepsilon > 2Mh$ and $Mh < L(\varepsilon)\varepsilon$, then

$$\begin{aligned} &\|x_{n+1} - y_{\ell(n+1)}\| - \|x_n - y_{\ell n}\| \\ &\quad \leq 2hL(\|x_n - y_{\ell n}\| + Mh) + Mh^2/\varepsilon \\ &\quad \leq 2hL(\|x_n - y_{\ell n}\| + Mh) + hL(\varepsilon) \\ &\quad \leq 3hL(\|x_n - y_{\ell n}\| + Mh). \end{aligned}$$

Lemma 2.2. *Let* A, x_0, *and* y_0 *be as in Lemma* 2.1. *Let* $T > 0$ *and* $T = nh_n$, *where* n *is a positive integer. Then for any* $\varepsilon > 0$ *there exist a positive number* $\delta = \delta(\varepsilon)$ *and a positive integer* $N = N(\varepsilon, x_0, y_0)$ *such that*

$$\left\| \left(1 - \frac{t}{n}A\right)^{-n} x_0 - \left(1 - \frac{t}{\ell n}A\right)^{-\ell n} y_0 \right\| \le \varepsilon$$

for $\|x_0 - y_0\| \le \delta$, $n \ge N$, $\ell = 1, 2, 3, \dots$, *and* $t \in [0, T]$.

Proof. Given $\varepsilon > 0$, choose $\eta = \eta(\varepsilon) > 0$ such that

$$\int_\eta^\varepsilon \frac{dy}{L(y)} \ge 4T.$$

Setting $\delta = \delta(\varepsilon) = \eta/3$, choose x_0, $y_0 \in D(A)$ such that $\|x_0 - y_0\| \le \delta$. Moreover, choose a positive integer $N = N(\varepsilon, x_0, y_0)$ such that $Mh_N = MT/N < \min(\eta/2, L(\varepsilon)\varepsilon)$. For simplicity, put

$$\alpha_k^n(t) = \left\| \left(1 - \frac{t}{n}A\right)^{-k} x_0 - \left(1 - \frac{t}{\ell n}A\right)^{-\ell k} y_0 \right\| \qquad \text{for } t \in [0, T].$$

Let $n \ge N$ and assume that $\alpha_n^n(t) > \varepsilon$ for some $t \in [0, T]$. Since $\alpha_0^n(t) = \|x_0 - y_0\| \le \eta/3$ and by noting (6) we have

$$\begin{aligned} \alpha_{n-1}^n &= \alpha_n^n(t) + \alpha_{n-1}^n(t) - \alpha_n^n(t) \ge \varepsilon - Mt/n \ge \varepsilon - Mh_N \\ &> \eta - \eta/2 = \eta/2, \end{aligned}$$

there exists a nonnegative integer $k_0 < k \le n$ such that

$$\alpha_{k_0}^n(t) < \eta/2 \quad \text{and} \quad \alpha_k^n(t) \ge \eta/2 \qquad \text{for } k_0 < k \le n.$$

For $t \in (0, T]$, putting $h = t/n$ we have

$$\begin{aligned} 4T &\le \int_\eta^\varepsilon \frac{dy}{L(y)} \le \int_{\alpha_{k_0}^n(t)+Mh}^{\alpha_n^n(t)+Mh} \frac{dy}{L(y)} \\ &\le \sum_{k=k_0}^{n-1} \frac{\alpha_{k+1}^n(t) + Mh - (\alpha_k^n(t) + Mh)}{L(\alpha_k^n(t) + Mh)} \\ &\le \sum_{k=k_0}^{n-1} 3h = 3(n - k_0)h \le 3t \le 3T, \qquad \text{by Lemma 2.1.} \end{aligned}$$

This is a contradiction. Hence $\alpha_n^n(t) \le \varepsilon$ for $n \ge N$ and $t \in [0, T]$.

Proof of Theorem 1. Given $\varepsilon > 0$, put $\delta(\varepsilon) = \varepsilon/2(|Au_0| + 1)$. Put $x_k^n = (1 - h_n A)^{-k} u_0$. Let n, $m \ge 2T|Au_0|/\varepsilon \equiv N_1(\varepsilon)$. In the case $t \in [0, \delta(\varepsilon)]$, taking the integer k_n so that $t \in ((k_n - 1)h_n, k_n h_n]$, we obtain

$$\begin{aligned}\|u_n(t) - u_0\| = \|x^n_{k_n} - u_0\| &= \left\|\sum_{j=1}^{k_n}(x^n_j - x^n_{j-1})\right\| \\ &\le \sum_{j=1}^{k_n}\|x^n_j - x^n_{j-1}\| \le \sum_{j=1}^{k_n} h_n|Au_0| \\ &\le k_n h_n|Au_0| \le (t + h_n)|Au_0| \\ &\le \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

Similarly we have

$$\|u_m(t) - u_0\| \le \varepsilon.$$

Thus

$$\|u_n(t) - u_m(t)\| \le \|u_n(t) - u_0\| + \|u_0 - u_m(t)\| \le \varepsilon + \varepsilon = 2\varepsilon$$

for $t \in [0, \delta(\varepsilon)]$ and $n, m \ge N_1(\varepsilon)$.

In the case $t \in [\delta(\varepsilon), T]$, by Lemma 2.2 there exists a positive integer $N_2(\varepsilon)$ such that

$$\left\|\left(1 - \frac{t}{k_n}A\right)^{-k_n} u_0 - \left(1 - \frac{t}{mk_n}A\right)^{-mk_n} u_0\right\| \le \varepsilon \quad \text{and} \quad h_n|Au_0| \le \varepsilon$$

for $t \in [\delta(\varepsilon), T]$ and $n, m \ge N_2(\varepsilon)$. Thus

$$\begin{aligned}\|u_n(t) - u_{nm}(t)\| &= \|(1 - h_n A)^{-k_n} u_0 - (1 - h_{nm}A)^{-k_{nm}} u_0\| \\ &\le \left\|\left(1 - \frac{k_n h_n}{k_n}A\right)^{-k_n} u_0 - \left(1 - \frac{k_n h_n}{mk_n}A\right)^{-mk_n} u_0\right\| \\ &\quad + \left\|(1 - h_{nm}A)^{-mk_n} u_0 - (1 - h_{nm}A)^{-k_{nm}} u_0\right\| \\ &\le \varepsilon + \|(1 - h_{nm}A)^{-mk_n} u_0 - (1 - h_{nm}A)^{-k_{nm}} u_0\| \\ &= \varepsilon + \left\|\sum_{j=k_{nm}+1}^{mk_n} (x^{nm}_j - x^{nm}_{j-1})\right\| \\ &\le \varepsilon + \sum_{j=k_{nm}+1}^{mk_n} \|x^{nm}_j - x^{nm}_{j-1}\| \\ &\le \varepsilon + mh_{nm}|Au_0| = \varepsilon + h_n|Au_0| \le 2\varepsilon\end{aligned}$$

for $t \in [\delta(\varepsilon), T]$ and $n, m \ge N_2(\varepsilon)$. Similarly we have

$$\|u_{nm}(t) - u_m(t)\| \le 2\varepsilon$$

for $t \in [\delta(\varepsilon), T]$ and $n, m \ge N_2(\varepsilon)$.

Putting $N(\varepsilon) = \max(N_1(\varepsilon), N_2(\varepsilon))$,

$$\|u_n(t) - u_m(t)\| \le 4\varepsilon$$

for $t \in [0, T]$ and n, $m \geq N(\varepsilon)$. Hence $\{u_n(t)\}$ converges uniformly on $[0, T]$. Put $u(t) = \lim_{n\to\infty} u_n(t)$ for $t \in [0, T]$. For $0 < s < t \leq T$, choose integers j, k such that $s \in ((j-1)h_n, jh_n]$ and $t \in ((k-1)h_n, kh_n]$. Then

$$\begin{aligned}\|u_n(t) - u_n(s)\| &= \|x_k^n - x_j^n\| \\ &\leq \sum_{i=j}^{k-1} \|x_{i+1}^n - x_i^n\| \leq \sum_{i=j}^{k-1} h_n |Au_0| \\ &\leq (k-j)h_n |Au_0| \leq (t - s - h_n)|Au_0|.\end{aligned}$$

Thus

$$\|u(t) - u(s)\| \leq (t-s)|Au_0| \qquad \text{as } n \to \infty.$$

This holds for $s = 0$. Hence,

$$\|u(t) - u(s)\| \leq |Au_0||t - s| \qquad \text{for all } t, s \in [0, T].$$

3. Generation of nonlinear semigroups

Let A be a directed L-dissipative operator in H satisfying the condition (R1). Following Benilan [1], we define an integral solution u to the Cauchy problem:

$$(CP_T; u_0)\begin{cases} (d/dt)u(t) \in Au(t) & \text{for } t \in [0, T), 0 < T \leq \infty, \\ u(0) = u_0 \end{cases}$$

as a continuous function on $[0, T)$ with $u(0) = u_0$ satisfying the inequality

$$\begin{aligned}(7) \qquad &\|u(t) - x_0\|^2 - \|u(s) - x_0\|^2 \\ &\leq \int_s^t \{\langle y_0, u(\tau) - x_0\rangle + L(\|u(\tau) - x_0\|)\|u(\tau) - x_0\|\}\, d\tau\end{aligned}$$

for $x_0 \in D(A)$, $y_0 \in Ax_0$ and $0 \leq s \leq t < T$.

It is easy to see that the limit function u as in Theorem 1 is an integral solution to $(CP_T; u_0)$, and satisfies

$$(8) \qquad \|v(t) - u(t)\| - \|v(s) - u(s)\| \leq \int_s^t L(\|v(\tau) - u(\tau)\|)\, d\tau,$$

$0 \leq s \leq t < T$, for any integral solution $v(t)$ to $(CP_T; v_0)$. Such a solution is called a mild solution of $(CP_T; u_0)$.

Lemma 3.1. *Let $u(t) = u(t, u_0)$ be the limit function as in Theorem 1, and $v(t) = v(t, v_0)$ be a mild solution to $(CP_T; v_0)$, then*

$$(9) \qquad \lim_{\|v_0 - u_0\| \to 0} \|v(t, v_0) - u(t, u_0)\| = 0 \qquad \textit{uniformly on } [0, T).$$

In particular, if $v_0 = u_0$ then $v(t) \equiv u(t)$.

Proof. Putting $s = 0$ in (8),

$$(10) \qquad \|v(t) - u(t)\| \leq \|v_0 - u_0\| + \int_0^t L(\|v(\tau) - u(\tau)\|)\, d\tau.$$

Putting $f(t) = \|v(t) - u(t)\|$ and $F(t) = \|v_0 - u_0\| + \int_0^t L(\|v(\tau) - u(\tau)\|)\,d\tau$, (10) is expressed as

$$f(t) \le F(t). \tag{10'}$$

Thus $F'(t) = L(f(t)) \le L(F(t))$. Hence,

$$\int_{\|v_0-u_0\|}^{F(t)} \frac{ds}{L(s)} = \int_0^t \frac{F'(\tau)}{L(F(\tau))}\,d\tau \le \int_0^t d\tau = t.$$

Since $\int_0^1 ds/L(s) = \infty$, $\lim_{\|v_0-u_0\|\to 0} F(t) = 0$ uniformly on $[0, T)$. So (9) is obtained.

Lemma 3.2. *Let* $u_0 \in \overline{D(A)}$. *Then there exists a unique mild solution to* $(CP_T; u_0)$.

Proof. Since $u_0 \in \overline{D(A)}$, there exists a sequence $\{x_n\} \subset D(A)$ which goes to u_0. Let $u(t, x_n)$ be the limit function with initial value x_n. By (9) we have

$$\lim_{n,m\to\infty} \|u(t, x_n) - u(t, x_m)\| = 0 \qquad \text{uniformly on } [0, T).$$

Putting $u(t) = \lim_{n\to\infty} u(t, x_n)$, it is clear that $u(t)$ is a mild solution to $(CP_T; u_0)$.

Next, letting $v(t)$ be any mild solution to $(CP_T; u_0)$, using (9) again,

$$\lim_{n\to\infty} \|v(t) - u(t, x_n)\| = 0.$$

Thus

$$\|v(t) - u(t)\| \le \|v(t) - u(t, x_n)\| + \|u(t, x_n) - u(t)\| \to 0$$

as $n \to \infty$. Hence, $v(t) \equiv u(t)$.

From Lemma 3.2, it is easily seen that there exists a unique mild solution $u(t, x)$ in $(CP_\infty; x)$ for any $x \in \overline{D(A)}$. Then we have the following theorem.

Theorem 2. *Let A be a directed L-dissipative operator satisfying the condition* (R1). *For each $x \in \overline{D(A)}$, let $u(t, x)$ be the unique mild solution to $(CP_\infty; x)$. Let $T(t)$ be an operator on $\overline{D(A)}$ such that $T(t)x = u(t, x)$. Then $\{T(t) | t \ge 0\}$ has the semigroup property.*

Proof. It is obvious that $T(t)x = x$ and $T(t+s)x = T(t)T(s)x$ for t, $s \ge 0$ and $x \in \overline{D(A)}$. We show that $u(t, x)$: $[0, \infty) \times \overline{D(A)} \to \overline{D(A)}$ is continuous. Assume that $(t_n, x_n) \to (t, x) \in [0, \infty) \times \overline{D(A)}$ and $(t_n, x_n) \to (t, x)$ as $n \to \infty$. Choose a sequence $\{x'_n\} \subset D(A)$ which converges to x. Then by (1),

$$\begin{aligned}
&\|u(t_n, x_n) - u(t, x)\| \\
&\quad \le \|u(t_n, x_n) - u(t_n, x'_n)\| + \|u(t_n, x'_n) - u(t_n, x'_m)\| \\
&\qquad + \|u(t_n, x'_m) - u(t, x'_m)\| + \|u(t, x'_m) - u(t, x)\| \\
&\quad \le \|u(t_n, x_n) - u(t_n, x'_n)\| + \|u(t_n, x'_n) - u(t_n, x'_m)\| \\
&\qquad + |Ax'_m||t_n - t| + \|u(t, x'_m) - u(t, x)\|.
\end{aligned}$$

By (9), for any $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that

$$\|u(t_n, x_n) - u(t_n, x_n')\| < \varepsilon/3,$$
$$\|u(t_n, x_n') - u(t_n, x_m')\| < \varepsilon/3,$$
$$\|u(t, x_m') - u(t, x)\| < \varepsilon/3$$

for n, $m \geq N$. Thus

$$\begin{aligned}\|u(t_n, x_n) - u(t, x)\| &\leq \varepsilon/3 + \varepsilon/3 + |Ax_m'||t_n - t| + \varepsilon/3 \\ &= \varepsilon + |Ax_m'||t_n - t|.\end{aligned}$$

Hence

$$\overline{\lim_{n\to\infty}} \|u(t_n, x_n) - u(t, x)\| \leq \varepsilon.$$

Remark. Let X be a real Banach space. In the case A is a continuous mapping from a subset of $[a, b) \times X$ $(a < b \leq \infty)$ into X, Iwamiya has given a result which guarantees existence and uniqueness of solutions under a weaker condition than (ii) in §1 (see [3] in detail) for the nonautonomous differential equation in X

$$(CP; \tau, z)\begin{cases} (d/dt)u(t) = A(t, u(t)), \\ u(t) = z, \end{cases}$$

giving (τ, z) in $[a, b) \times X$.

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