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A REMARK ON FINITELY PRESENTED INFINITE DIMENSIONAL ALGEBRAS

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ABSTRACT. By estimating dimensions of representation varieties, we show that certain finitely presented algebras are infinite dimensional.

The object of this note is to use a tiny sliver of the geometry of representations of a finitely generated algebra to prove the following

Theorem. *Let A be an augmented algebra (over a field k) with augmentation ideal $\mathscr{A}$, given by the finite presentation*

$$A = \langle x_1, \cdots, x_m; w_1^q, \cdots, w_n^q \rangle \qquad (w_i \in \mathscr{A}, q \geq 2).$$

If

$$n \leq (m-1)q$$

then A is infinite dimensional.

Our theorem should be viewed in the light of the following theorem of J. Levitzki [3]: *if every element of the augmentation ideal of a finitely generated augmented algebra A is nilpotent of bounded degree, then A is finite dimensional.* It also can be viewed as a counterpart to a similar theorem about finitely presented groups that we proved a couple of years ago [1].

Before we embark on the proof of the theorem we observe first that we lose nothing if we assume that k is algebraically closed. The basic idea is to make use of the parametrisation of the set $X(A, q)$ of all the equivalence classes of semisimple representations of A in $M(q, k)$, the k-algebra of all $q \times q$ matrices over k, given by Procesi [4]. We recall the details of this parametrisation in a form suitable for the purposes we have in mind. To this end, suppose that F is the free associative k-algebra on $x_1, \ldots, x_m$, and that $U = M(q, k)^m$. We associate to each representation ρ of F the point

$$u = (\rho(x_1), \ldots, \rho(x_m)) \in U.$$

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Each element $w = w(x_1, \dots, x_m) \in F$ defines q polynomial functions f_w^i $(i = 0, \cdots, q-1)$ on U, as follows:

> $f_w^i(u) =$ *the coefficient of the degree* i *term of the characteristic polynomial of* $\rho(w)$, *where* ρ *is the representation of* F *in* $M(q, k)$ *corresponding to* u.

Then it turns out that the k-subalgebra B of F generated by these polynomial functions f_w^i, where w ranges over F and $i = 0, \dots, q-1$, is a finitely generated subalgebra of the k-algebra P of all polynomial functions on U. Notice that P is the k-algebra of polynomials in mq^2 variables. Now let $X(F, q)$ be the affine algebraic set corresponding to this algebra B and let p be the canonical map from U into $X(F, q)$. Then Procesi [4] proves that the following hold:

(1) p is onto $X(F, q)$;

(2) if S is the subset of U consisting of semi-simple representations of F, then p maps S onto $X(F, q)$;

(3) if $\rho \in S$ is irreducible, then $p^{-1}(p(\rho))$ is the set of all representations of F in $M(m, q)$ equivalent to ρ.

It follows, in particular, that if ρ is an irreducible representation of F, then $p^{-1}(\rho)$ is of dimension $q^2 - 1$. Now $X(F, 2)$ is an affine variety, i.e. it is irreducible. So it follows (see e.g. Humphreys [2, page 30]) that

$$\begin{aligned}\dim(X(F, q)) &\geq \dim(R(F, q)) - \dim(p^{-1}(\rho)) \\ &= mq^2 - (q^2 - 1) = (m-1)q^2 + 1.\end{aligned}$$

Now if M is a matrix of degree q over k, then $M^q = 0$ if and only if its characteristic polynomial is t^q. This means that the coefficients of all of the powers of t except for t^q in the characteristic polynomial of M are zero. We need to apply this remark to the defining relations of A. Observe then that $\rho(w_j^q) = 0$ for every representation ρ of A in $M(q, k)$ if and only if the functions $f_{w^j}^i = 0$, for $i = 0, \cdots, q-1$. The existence of at least one such representation is guaranteed by the hypothesis since A is an augmented algebra. Consider then these functions $f_{w^j}^i = g_j^i$ $(i = 0, \cdots, q-1, j = 1, \cdots, n)$. Every such function g_j^i lies in the algebra B. Consequently they can be viewed as polynomial functions on $X(F, q)$ with values in k. Let $h_j^i = p \circ f_j^i$. Consider

$$V = \bigcap_{j,i} (h_j^i)^{-1}(0).$$

This is therefore an affine algebraic set, since the h_j^i are polynomial functions. And

$$p(V) = \bigcap_{j,i} (g_j^i)^{-1}(0)$$

is therefore also an affine algebraic set in $X(F, d)$ whose dimension is then bounded as follows:

$$\dim p(V) \geq (m-1)q^2 + 1 - nq .$$

Now every point in $p(V)$ corresponds to an equivalence class of semi-simple representations of F such that for each representation $\rho \in V$ the characteristic polynomial of every $\rho(w_j)$ is t^q. So by the remark above, ρ factors through A, yielding a semi-simple representation of A. This means that $p(V)$ parametrises a set of inequivalent semi-simple representations of A. Since a finite dimensional algebra has only finitely many inequivalent representations overall, if $\dim p(V) > 0$ then A is infinite dimensional. This is precisely what the inequality given in the theorem ensures.

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