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FIXED POINT THEOREMS IN PRODUCT SPACES

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ABSTRACT. If K_1 and K_2 are nonempty closed weakly compact subsets of Banach spaces and they have the generic fixed point property for nonexpansive mappings, then in the maximum norm $K_1 \times K_2$ has fixed point property for nonexpansive mappings.

In this paper we study fixed points of nonexpansive mappings in product spaces.

Let (K, ζ_1) and (L, ζ_2) be metric spaces. A mapping $T: K \to L$ is nonexpansive if

$$\zeta_2(Tx, Ty) \le \zeta_1(x, y)$$

for all $x, y \in K$. Let χ denote the Hausdorff (or ball) measure of noncompactness in K and L. A continuous mapping $T: K \to L$ is a k-set contraction $k \in [0, 1))$, if for any $K_0 \subset K$ we have

$$\chi(TK_0) \le k\chi(K_0).$$

(K, ζ_1) has a fixed point property for nonexpansive (continuous) mappings if every nonexpansive (continuous) self-mapping $T: K \to K$ must have a fixed point.

Let K be a nonempty, convex, weakly compact subset of a Banach space X. K has the generic fixed point property for nonexpansive mappings if for every nonexpansive self-mapping $T: K \to K$ and every nonempty, convex, closed subset $K_0 \subset K$ with $TK_0 \subset K_0$, we have $K_0 \cap \mathrm{Fix}(T) \neq \varnothing$, where $\mathrm{Fix}(T) = \{x \in K: Tx = x\}$.

Suppose that $(X, \|\ \|_1)$ is a Banach space and (K_2, ζ_2) is a metric space. Let $X \times K_2$ denote a product space of X and K_2 with a maximum metric

$$\zeta_\infty((x, u), (y, v)) = \max(\|x - y\|_1, \zeta_2(u, v))$$

for $x, y \in X$ and $u, v \in K_2$. It was shown in [7] (see also [2, 5, 6, 8, 9, 10]) that if X has KK-norm, $\varnothing \neq K_1 \subset X$ is weakly compact and convex, and K_1 and K_2 have the fixed point property for nonexpansive mappings, then every

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nonexpansive (with respect to the metric ζ_∞) $T\colon K_1 \times K_2 \to K_1 \times K_2$ has a fixed point. Here we generalize this result. We begin with

Theorem 1. *Let $(X, \|\ \|_1)$ be a Banach space and let (K_2, ζ_2) be a metric space. Suppose that $\varnothing \neq K_1 \subset X$ is weakly compact, convex and has the generic fixed point property for nonexpansive mappings. If $F\colon K_1 \times K_2 \to K_1$ is a nonexpansive mapping, then there exists a nonexpansive mapping $R\colon K_1 \times K_2 \to K_1$ such that*

$$F(R(x,u),u) = R(x,u)$$

for $(x,u) \in K_1 \times K_2$ and $R(x,u) = x$ when $F(x,u) = x$.

Proof. We construct a mapping $\tilde{F}\colon K_1 \times K_2 \to K_1 \times K_2$ in the following way

$$\tilde{F}(x,u) = (F(x,u),u) = (F, P_2)(x,u)$$

for $(x,u) \in K_1 \times K_2$, where P_2 is the second coordinate projection. It is trivial to check that $\tilde{F}$ is nonexpansive and $A = \mathrm{Fix}(\tilde{F}) \neq \varnothing$. Now we define the set

$N(A) = \{\tilde{G} = (G, P_2) \in K_1 \times K_2^{K_1 \times K_2}\colon G\colon K_1 \times K_2 \to K_1$ is nonexpansive and $A \subset \mathrm{Fix}(\tilde{G})\}$.

Note that

$$N(A) \subset \prod_{(x,u)\in K_1\times K_2} K_1 \times \{u\} \sim \prod_{(x,u)\in K_1\times K_2} K_1.$$

In K_1 we have the weak topology, so by Tychonoff's Theorem $\prod_{(x,u)\in K_1\times K_2} K_1$ is compact in the product topology. This topology can be transposed on the set $\prod_{(x,u)\in K_1\times K_2} K_1 \times \{u\}$ and $N(A)$ is closed in this topology. It allows us to apply Bruck's method ([1]) step-by-step to obtain a nonexpansive retraction $r\colon K_1 \times K_2 \to A$ which additionally belongs to $N(A)$, i.e. $r = (R, P_2)$. R satisfies the desired conditions.

Let X, K_1, K_2 be as in Theorem 1. Suppose that $T = (T_1, T_2)\colon K_1 \times K_2 \to K_1 \times K_2$ is a mapping such that $T_1\colon K_1 \times K_2 \to K_1$ is nonexpansive. Fix $x_0 \in K_1$. The mapping $\overline{T}_2\colon K_2 \to K_2$ is defined by

$$\overline{T}_2(u) = T_2(R(x_0,u),u), \qquad u \in K_2,$$

where R satisfies Theorem 1. It is clear that $\overline{T}_2(u) = u$ implies that $T(R(x_0,u),u) = (R(x_0,u),u)$. Let K_2 have the fixed point property with respect to a subclass S of $K_2^{K_2}$. If $\overline{T}_2 \in S$, then $\mathrm{Fix}(T) \neq \varnothing$.

Theorem 2. *Let X, K_1, K_2 be as in Theorem 1 and let $T_1\colon K_1 \times K_2 \to K_1$ be nonexpansive.*

(i) *If (K_2, ζ_2) has a fixed point property for nonexpansive mappings and $T_2\colon K_1 \times K_2 \to K_2$ is nonexpansive, then $T = (T_1, T_2)\colon K_1 \times K_2 \to K_1 \times K_2$ has a fixed point.*

(ii) *If (K_2, ζ_2) has a fixed point property for continuous mappings and $T_2\colon K_1 \times K_2 \to K_2$ is continuous, then $T = (T_1, T_2)\colon K_1 \times K_2 \to K_1 \times K_2$ has a fixed point.*

(iii) *If* $\varnothing \neq K_2$ *is a convex, closed subset of a Banach space* X *and* $T_2\colon K_1 \times K_2 \to K_2$ *is a* k*-set contraction with respect to Hausdorff measure of noncompactness, then* $T = (T_1, T_2)\colon K_1 \times K_2 \to K_1 \times K_2$ *has a fixed point.*

Proof. It is sufficient to observe that in the first case S is the set of all nonexpansive self-mappings in K_2, in the second one S is the set of all continuous self-mappings in K_2, and in the third one S is the set of all k-set contractions in K_2.

Remark 1. If $K_1, \dots, K_n$ are nonempty, convex, weakly compact subsets of Banach spaces and have the generic fixed point property for nonexpansive mappings, then $\prod_{i=1}^n K_i$ has a fixed point property for nonexpansive mappings (with respect to the maximum norm).

Remark 2. For discussion of spaces which have the fixed point property with respect to nonexpansive mappings, we refer to Kirk and ([3, 4]) and Reich ([11, 12]).

Remark 3. If X_1 is the conjugate Banach space, then in the above theorems the weak topology can be replaced by the weak-$*$ topology.

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