PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 108, Number 3, March 1990

A CHARACTERIZATION OF THE MACKEY TOPOLOGY $\tau(L^{\infty}, L^{1})$

MARIAN NOWAK

(Communicated by John B. Conway)

ABSTRACT. We give a description of the Mackey topology $\tau(L^{\infty}, L^{1})$ for finite measures in terms of a family of norms defined by certain Young functions. As an application we obtain various topological characterizations of sequential convergence in $\tau(L^{\infty}, L^{1})$. Moreover, we obtain a criterion for relative weak compactness in L^{1} in terms of the integral functional defined by some Young function.

1. INTRODUCTION

J. B. Cooper [3, Chapter III] has characterized the Mackey topology $\tau(L^{\infty}, L^{1})$ on L^{∞}-space associated with a positive Radon measure on a locally compact space, in terms of the notion of mixed topologies. K. D. Stroyan [10] has characterized $\tau(L^{\infty}, L^{1})$ for finite measures in terms of an infinitesimal relation on the nonstandard extension ${}^{*}L^{\infty}$. In [9] we examined the topology $\tau(L^{\infty}, L^{1})$ from the viewpoint of the theory of locally solid Riesz spaces (see [1]).

Let (Ω, Σ, μ) be a σ-finite measure space, and let L^{0} denote the set of equivalence classes of all real-valued μ-measurable functions defined and finite a.e. on Ω. Then L^{0} is a super Dedekind complete Riesz space under the ordering $x \leq y$ whenever $x(t) \leq y(t)$ a.e. on Ω. The Riesz F-norm

$$\|x\|_{0} = \int_{\Omega} |x(t)|(1 + |x(t)|)^{-1} f(t)\, d\mu \qquad \text{for } x \in L^{0}$$

where a function $f: \Omega \to (0, \infty)$ is μ-measurable with $\int_{\Omega} f(t)\, d\mu = 1$, determines a Lebesgue topology $\mathcal{T}_{0}$ (see [5, Chapter I, §6], [1, Theorem 24.7]). This topology generates convergence in measure on every measurable set of finite measure. Let L^{∞} denote the set of all $x \in L^{0}$ such that $\|x\|_{\infty} = \operatorname{ess\,sup}_{t \in \Omega} |x(t)| < \infty$ and let $\mathcal{T}_{\infty}$ denote the topology of the B-norm $\| \ \|_{\infty}$.

Received by the editors June 17, 1988. The results of this paper were presented on August 31, 1989 at the 2nd International Conference on Function Spaces, A. Mickiewicz University, Poznan (Poland).

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46E30.

Key words and phrases. Mackey topology $\tau(L^{\infty}, L^{1})$, Orlicz spaces, mixed topologies, locally solid Riesz spaces.

The absolute weak topology $|\sigma|(L^\infty, L^1)$ is a locally convex-solid topology on L^∞ defined by the family of Riesz seminorms $[\rho_y : y \in L^1]$ where

$$\rho_y(x) = \int_\Omega |x(t)y(t)|\, d\mu \qquad \text{for } x \in L^\infty .$$

It is known that $|\sigma|(L^\infty, L^1)$ is a Lebesgue topology (see [1, Theorem 6.6 and Theorem 9.1]).

In [9] we showed that $\tau(L^\infty, L^1)$ is the finest Hausdorff Lebesgue topology on L^∞, and that it coincides with the mixed topology $\gamma[\mathscr{T}_\infty, \mathscr{T}_{0|L^\infty}]$. For terminology concerning mixed topologies we refer to [11]. Note that in view of the Amemiya theorem [1, Theorem 12.9] $\mathscr{T}_0$ and $|\sigma|(L^\infty, L^1)$ induce the same topology on $\| \; \|_\infty$-bounded subsets of L^∞, and therefore, the mixed topologies $\gamma[\mathscr{T}_\infty, \mathscr{T}_{0|L^\infty}]$ and $\gamma[\mathscr{T}_\infty, |\sigma|(L^\infty, L^1)]$ coincide (see [11, Theorem 2.2.2]). Moreover, if the measure μ is finite then, by the same argument, $\gamma[\mathscr{T}_\infty, \mathscr{T}_{0|L^\infty}]$ coincides also with the mixed topology $\gamma[\mathscr{T}_\infty, \mathscr{T}_{p|L^\infty}] (1 \le p < \infty)$, where $\mathscr{T}_p$ denotes the $\| \; \|_p$-norm topology on L^p.

By a Young function we mean a function $\varphi : [0, \infty) \to [0, \infty]$ which is convex, left continuous, continuous at zero with $\varphi(0) = 0$, not identically equal to zero. We denote by L^φ the Orlicz space associated with a Young function φ (see [6, 7] for more details). Note that this includes L^φ being equal to L^∞ and L^1. Let $\| \cdot \|_\varphi$ and $\| \cdot \|_\varphi^0$ denote the Luxemburg B-norm and the Orlicz B-norm defined on L^φ by (see [6, 7]):

$$\|x\|_\varphi = \inf \left\{ \lambda > 0 : \int_\Omega \varphi(|x(t)|/\lambda)\, d\mu \le 1 \right\}$$

$$\|x\|_\varphi^0 = \sup \left\{ \left| \int_\Omega x(t)y(t)\, d\mu \right| : y \in L^{\varphi^*},\ \|y\|_{\varphi^*} \le 1 \right\}$$

where φ^* denotes the function complementary to φ in the sense of Young, i.e. $\varphi^*(v) = \sup\{uv - \varphi(u) : u \ge 0\}$ for $v \ge 0$. We shall use the following inequalities (see [6, p. 80; 7, p. 48])

$$(+) \qquad \|x\|_\varphi \le \|x\|_\varphi^0 \le 2\|x\|_\varphi \qquad \text{for } x \in L^\varphi .$$

For $r > 0$ we will write $B_\varphi(r) = \{x \in L^\varphi : \|x\|_\varphi \le r\}$.

A Young function φ is called an N-function if it takes only finite values, vanishes only at zero and $\varphi(u)/u \to 0$ as $u \to 0$, $\varphi(u)/u \to \infty$ as $u \to \infty$ (see [6, p. 9]).

The following lemma will be needed.

Lemma 1.1. *Let φ be an N-function. Then there exists an N-function ψ satisfying the Δ_2-condition (i.e. $\limsup_{u\to\infty}(\psi(2u)/\psi(u)) < \infty$) and such that $\psi(u) \le \varphi(u)$ for $u \ge 0$.*

Proof. In view of [6, p. 6] $\varphi(u) = \int_0^u p(t)\, dt$ for $u \ge 0$, where the function $p(t)$ is right continuous for $t \ge 0$, nondecreasing and such that $p(0) = 0$, $p(t) > 0$

for $t > 0$ and $p(t) \to \infty$ as $t \to \infty$. Let us put

$$q(t) = \begin{cases} p(t) \text{ if } 0 \le t < 1, \\ \min(2^k p(2^{n-k-1}) \colon k = 0, 1, \dots, n-1) \text{ if } 2^{n-1} \le t < 2^n, \end{cases}$$

and define

$$\psi(u) = \int_0^u q(t)\,dt \qquad \text{for } u \ge 0.$$

It is seen that ψ is an N-function, and that $q(t) = \min(2q(2^{n-2}),\ p(2^{n-1}))$ for $t \in [2^{n-1}, 2^n)$, $n = 2, 3, \dots$. Thus $\psi(u) \le \varphi(u)$ for $u \ge 0$. We shall show that ψ satisfies the Δ_2-condition. Indeed, let $t \ge 1$. Choose a natural number n such that $t \in [2^{n-1}, 2^n)$. Then we have

$$\frac{q(2t)}{q(t)} = \frac{q(2^n)}{q(2^{n-1})} \le \frac{2q(2^{n-1})}{q(2^{n-1})} = 2.$$

Hence for $u \ge 2$ we get

$$\psi(2u) \le 2uq(2u) \le 4uq(u) \le 8uq(u/2) = 16(u/2)q(u/2) \le 16\psi(u),$$

and this means that ψ satisfies the Δ_2-condition.

We denote by χ_E the characteristic function of the subset E of Ω.

Henceforth we will assume that the measure μ is finite.

2. A characterization of the Mackey topology $\tau(L^\infty, L^1)$ for finite measures

We start by giving a characterization of absolutely continuous seminorms on L^∞. Note that the Riesz seminorm ρ on L^∞ is absolutely continuous iff $\rho(\chi_E) \to 0$ as $\mu(E) \to 0$. It is known that ρ is absolutely continuous iff it is order continuous (i.e. $\rho(x_n) \to 0$ if $x_n \downarrow 0$ holds in L^∞) (see [8, Theorem 2.1]).

We will write

$$B_\infty = \{x \in L^\infty \colon \|x\|_\infty \le 1\} \qquad \text{and} \qquad B_\rho = \{x \in L^\infty \colon \rho(x) \le 1\}.$$

Proposition 2.1. *For a Riesz seminorm ρ on L^∞ the following statements are equivalent*:

(i) *ρ is absolutely continuous.*

(ii) *There exists an N-function φ such that $\rho(x) \le \|x\|_\varphi$ for $x \in L^\infty$.*

Proof. (i) $\Rightarrow$ (ii) For $y \in L^1$ let us put $f_y(z) = \int_\Omega z(t)y(t)\,d\mu$ for $z \in L^\infty$. Denoting by $(L^\infty, \rho)^*$ the topological dual of (L^∞, ρ) we have $(L^\infty, \rho)^* \subset \{f_y \colon y \in L^1\}$, because ρ is order continuous on L^∞ and L^1 is the Köthe dual of L^∞. Using the Hahn-Banach theorem for the seminormed space (L^∞, ρ), for $x \in L^\infty$ we get

$$\rho(x) = \sup\{|f_y(x)| \colon f_y \in (L^\infty, \rho)^*, \|f_y\|_\rho \le 1\}$$

where $\|f_y\|_\rho = \sup\{|f_y(z)|: z \in B_\rho\}$. Therefore, writing

$$Y = \{y \in L^1: |f_y(z)| \le 1 \text{ for } z \in B_\rho\}$$

we get

$$\rho(x) = \sup_{y\in Y} \left| \int_\Omega x(t)y(t)\, d\mu \right| . \tag{1}$$

There exists a number $c > 0$ such that $\rho(z) \le c\|z\|_\infty$ for $z \in L^\infty$ (see [1, Theorem 16.7]). Applying $(+)$, for $y \in L^1$ we get

$$\begin{aligned}\int_\Omega |y(t)|\, d\mu &\le \sup\left\{\left|\int_\Omega z(t)y(t)\, d\mu\right| : z \in B_\infty\right\} \\ &\le c \sup\left\{\left|\int_\Omega z(t)y(t)\, d\mu\right| : z \in B_\rho\right\} .\end{aligned}$$

Hence

$$\sup_{y\in Y} \int_\Omega |y(t)|\, d\mu \le c . \tag{2}$$

For a measurable subset E of Ω, by (1), we have $\rho(\chi_E) = \sup_{y\in Y} \int_E |y(t)|\, d\mu$. Therefore, there exists a sequence of positive numbers (λ_n) such that $\lambda_n \downarrow 0$ and

$$\sup_{y\in Y} \int_E |y(t)|\, d\mu \le 2^{-2n-1} \qquad \text{if } \mu(E) \le \lambda_n . \tag{3}$$

Choose a natural number n_0 such that $2^{-n_0} \le \lambda_1 2^{-3} c^{-1}(\mu(\Omega))^{-1}$ and set

$$q(s) = \begin{cases} \lambda_1 c^{-1} 2^{-n_0-1} s & \text{if} \quad 0 \le s < 2c\lambda_1^{-1} , \\ 2^{n-n_0-1} & \text{if} \quad 2c\lambda_n^{-1} \le s < 2c\lambda_{n+1}^{-1} ,\ n = 1, 2, \dots . \end{cases}$$

Define

$$\psi(u) = \int_0^u q(s)\, ds \qquad \text{for } u \ge 0 .$$

Then ψ is an N-function and $\psi(2u) \le 2^{n-n_0} u$ for $u \in [c\lambda_n^{-1}, c\lambda_{n+1}^{-1})$, $n = 1, 2, \dots$. For $y \in L^1$ let us write

$$\begin{aligned} E_0(y) &= \{t \in \Omega: |y(t)| < c\lambda_1^{-1}\} , \\ E_n(y) &= \{t \in \Omega: c\lambda_n^{-1} \le |y(t)| < c\lambda_{n-1}^{-1}\} ,\ n = 1, 2, \dots . \end{aligned}$$

Then by (2), for $y \in Y$ we have $\mu(E_n(y))c\lambda_n^{-1} \le \int_\Omega |y(t)|\, d\mu \le c$, so $\mu(E_n(y)) \le \lambda_n$. Therefore, according to (3), for $y \in Y$ we get

$$\begin{aligned} \int_\Omega \psi(2|y(t)|)\, d\mu &= \int_{E_0(y)} \psi(2|y(t)|)\, d\mu + \sum_{n=1}^\infty \int_{E_n(y)} \psi(2|y(t)|)\, d\mu \\ &\le \psi(2c\lambda_1^{-1})\mu(\Omega) + \sum_{n=1}^\infty 2^{n-n_0} \int_{E_n(y)} |y(t)|\, d\mu \le 2^{-1} . \end{aligned}$$

Thus $\|y\|_\psi \le 2^{-1}$ if $y \in Y$, and by (1) and (+), for $x \in L^\infty$ we get

$$\begin{aligned}\rho(x) &= \sup_{y\in Y}\left|\int_\Omega x(t)y(t)\,d\mu\right| \\ &\le \sup\left\{\left|\int_\Omega x(t)y(t)\,d\mu\right| : y \in L^\psi,\ \|y\|_\psi \le 2^{-1}\right\} \le \|x\|_{\psi^*}.\end{aligned}$$

Thus, to finish the proof it suffices to put $\varphi = \psi^*$.

(ii) $\Rightarrow$ (i) It suffices to show that for an N-function φ the norm $\|\cdot\|_\varphi$ is absolutely continuous on L^∞. Indeed, for a measurable subset E of Ω we have $\|\chi_E\|_\varphi = 1/\varphi^{-1}(\mu(E))^{-1})$ (see [6, p. 79]), and thus $\|\chi_E\|_\varphi \to 0$ as $\mu(E) \to 0$.

Thus the proof is finished.

Remark. The above result is motivated by Andô's paper [2] where a description of absolutely continuous seminorms on Orlicz spaces L^φ defined by some finite-valued Young function is given.

We are now ready to state our main result.

Theorem 2.2. *Let Φ_N be the collection of all N-functions. Then the Mackey topology $\tau(L^\infty, L^1)$ is generated by the family $\{\|\cdot\|_{\varphi|L^\infty} : \varphi \in \Phi_N\}$.*

Proof. We know that $\tau(L^\infty, L^1)$ is the finest Hausdorff Lebesgue topology on L^∞. Denote by τ_* the topology on L^∞ generated by the family $\{\|\cdot\|_{\varphi|L^\infty} : \varphi \in \Phi_N\}$. According to Proposition 2.1, for each $\varphi \in \Phi_N$ the norm $\|\cdot\|_\varphi$ is absolutely continuous on L^∞, so τ_* is a Lebesgue topology, and therefore, $\tau_* \subset \tau(L^\infty, L^1)$. Since $\tau(L^\infty, L^1)$ is a locally convex-solid topology, there exists a family $\{\rho_\alpha\}$ of Riesz seminorms on L^∞ that generates $\tau(L^\infty, L^1)$. Each ρ_α is absolutely continuous, and according to Proposition 2 we obtain $\tau(L^\infty, L^1) \subset \tau_*$.

The next theorem characterizes sequential convergence in $(L^\infty, \tau(L^\infty, L^1))$.

Theorem 2.3. *For a sequence (x_n) in L^∞ the following statements are equivalent:*

(i) $x_n \to 0$ *for* $\tau(L^\infty, L^1)$.
(ii) $\|x_n\|_\varphi \to 0$ *for each N-function* φ.
(iii) $\|x_n\|_0 \to 0$ *and* $\sup_n \|x_n\|_\infty < \infty$.
(iv) $\int_\Omega |x_n(t)y(t)|\,d\mu \to 0$ *for every* $y \in L^1$.
(v) $\|x_n\|_p \to 0$ *for some* $1 \le p < \infty$ *and* $\sup_n \|x_n\|_\infty < \infty$.

Proof. (i) $\Leftrightarrow$ (ii) It follows from Theorem 2.2.

(i) $\Leftrightarrow$ (iii) See [9, Theorem 1].

(i) $\Leftrightarrow$ (iv) See [9, Theorem 9].

(iii) $\Leftrightarrow$ (v) In view of [1, Theorem 12.9] $\mathscr{T}_p$ coincides with $\mathscr{T}_0$ on $\|\cdot\|_\infty$-bounded sets in L^∞.

3. Weak compactness in L^1

In this section we use Theorem 1.2 to obtain some interesting criterion for relative weak compactness in L^1. This criterion is analogous to the Andô's criterion for relative $\sigma(L^\varphi, L^{\varphi^*})$-compactness in Orlicz spaces L^φ associated with finite-valued Young functions φ such that $\varphi(u)/u \to \infty$ as $u \to \infty$ (see [3, Theorem 2]).

Theorem 3.1. *For a subset A of L^1 the following statements are equivalent:*

(i) *A is relatively compact for the weak topology $\sigma(L^1, L^\infty)$.*

(ii) *There exists an N-function φ such that*

$$\sup_{x\in A}\int_\Omega \varphi(|x(t)|)\,d\mu < \infty.$$

Proof. (i) $\Rightarrow$ (ii) Since $\tau(L^\infty, L^1)$ is the topology of uniform convergence on the weakly compact subsets of L^1, in view of Theorem 2.2 there exist an N-function ψ and a number $r > 0$ such that $B_\psi(r) \cap L^\infty \subset A^0$, where A^0 denotes the polar of A with respect to the dual pair (L^∞, L^1). Hence

$$\begin{aligned} A \subset A^{00} &\subset (B_\psi(r) \cap L^\infty)^0 \\ &= \left\{ x \in L^1 : \left| \int_\Omega x(t)y(t)\,d\omega \right| \le r^{-1} \quad \text{for } y \in B_\psi(1) \cap L^0 \right\}. \end{aligned}$$

According to the argument from [6, pp. 86–87] we have the following representation of the Orlicz B-norm $\|.\|^0_{\psi^*}$ on L^{ψ^*}: $\|x\|^0_{\psi^*} = \sup\{|\int_\Omega x(t)y(t)\,d\mu| : y \in B_\psi(1) \cap L^\infty\}$. Therefore, $\sup_{x\in A}\|x\|_{\psi^*} \le r^{-1}$, because $\|x\|_{\psi^*} \le \|x\|^0_{\psi^*}$ for all $x \in L^{\psi^*}$. In view of Lemma 1.1 there exists an N-function φ satisfying the Δ_2-condition and such that $\varphi(u) \le \psi^*(u)$ for $u \ge 0$. Thus $\sup_{x\in A}\|x\|_\varphi \le r^{-1}$ and hence [6, p. 77] $\sup_{x\in A}\int_\Omega \varphi(|x(t)|)\,d\mu < \infty$.

(ii) $\Rightarrow$ (i) Applying the Hölder inequality [6, Theorem 9.3], for a measurable subset E of Ω we have

$$\sup_{x\in A}\int_\Omega |x(t)\chi_E(t)|\,d\mu \le \|\chi_E\|^0_{\varphi^*} \sup_{x\in A}\|x\|_\varphi .$$

Since $\sup_{x\in A}\|x\|_\varphi < \infty$ and $L^\infty \subset (L^{\varphi^*})_a$ (= the ideal of elements of absolutely continuous norm in L^{φ^*}) we get $\sup_{x\in A}\int_E |x(t)|\,d\mu \to 0$ as $\mu(E) \to 0$. It is seen that the set A is $\|.\|_1$-bounded. Therefore, according to the Dunford-Pettis theorem (on relatively weakly compact subsets of L^1) (see [4, p. 294]) we obtain that A is relatively compact for $\sigma(L^1, L^\infty)$.

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INSTITUTE OF MATHEMATICS, A. MICKIEWICZ UNIVERSITY, 60-769 POZNAN, POLAND