

CONTINUOUS RINGS WITH ACC ON ESSENTIALS ARE ARTINIAN

S. K. JAIN, S. R. LOPEZ-PERMOUTH, AND S. TARIQ RIZVI

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ABSTRACT. It is proved that a left continuous ring with ascending chain condition on essential left ideals is left artinian.

1. INTRODUCTION

It is well known that for left or right self-injective rings diverse chain conditions on certain classes of left ideals are actually equivalent to the minimum condition on the lattice of all left ideals. Recently, Huynh-Dung and Page-Yousif proved independently that a left or right self-injective ring with ascending chain condition on essential left ideals is left artinian. In this paper we prove the preceding result by assuming that R is left continuous instead of being left self-injective. Examples are given to show that one-sided continuity and one-sided chain conditions may not necessarily yield the continuity or chain conditions on the opposite sides.

2. PRELIMINARIES

As defined by Utumi, a ring R is called left continuous if (i) every left ideal of R is essential in a direct summand of R and (ii) every left ideal isomorphic to a direct summand of R is itself a direct summand [6]. Continuous modules are defined analogously. Throughout, all rings have identity element and all modules are unital. $Z({}_R R)$, $J(R)$, and $\text{Soc}({}_R R)$ will denote, respectively, the left singular ideal, the Jacobson radical, and the left socle of a ring R .

We record below some well-known results often referred to in the proof of our theorem.

Lemma 1. *If R has acc on essential left ideals, then $Z({}_R R)$ is nilpotent.*

Proof. The proof is exactly the same as the one when R has acc on left annihilators ([5, Lemma 3.39, p. 380]).

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Lemma 2. *Let R be a left continuous ring. Then $J(R) = Z({}_R R)$, $R/J(R)$ is a von Neumann regular left continuous ring, and idempotents modulo $J(R)$ can be lifted.*

Proof. See Utumi [6].

The following lemma has recently been proved independently by Huynh-Dung and Page-Yousif. For completeness we include here its proof by Huynh-Dung [2].

Lemma 3. *For each left module M , M has acc on essential submodules if and only if $M/\text{Soc}(M)$ is left noetherian.*

Proof. If M has acc on essential submodules, each submodule N of M has the same property. Therefore, if $B \subset' N \subset M$, N/B is noetherian. In particular, it follows that every uniform submodule of M is noetherian. Consider a complement H of $\text{Soc}(M)$. Then $\text{Soc}(M) \oplus H \subset' M$, and therefore $\frac{M}{\text{Soc}(M) \oplus H}$ is noetherian. Since

$$\frac{M}{\text{Soc}(M) \oplus H} \simeq \frac{\frac{M}{\text{Soc}(M)}}{\frac{\text{Soc}(M) \oplus H}{\text{Soc}(M)}} \quad \text{and} \quad \frac{\text{Soc}(M) \oplus H}{\text{Soc}(M)} \simeq H,$$

it is now sufficient to show that H is noetherian in order to obtain that $\frac{M}{\text{Soc}(M)}$ is noetherian. Let $X = X_1 \oplus X_2 \oplus \cdots$ be a direct sum of nonzero submodules contained in H . Since for each i , $\text{Soc}(M) \cap X_i = 0$, it follows that each X_i contains a proper essential submodule Y_i . Thus $Y = Y_1 \oplus Y_2 \oplus \cdots$ is an essential submodule of X , and therefore $\frac{X}{Y} \simeq \frac{X_1}{Y_1} \oplus \frac{X_2}{Y_2} \oplus \cdots$ is noetherian. Consequently, the sum $X = X_1 \oplus X_2 \oplus \cdots$ must be a finite sum, and therefore H has finite Goldie dimension, say, k . Let $U = U_1 \oplus \cdots \oplus U_k$ be a direct sum of uniform submodules which is essential in H . Since H/U and U are noetherian, H is noetherian. This proves the claim. The converse is trivial.

Lemma 4. *Let $M = \bigoplus \sum_{i=1}^n A_i$. Then M is continuous if and only if each A_i is continuous and A_j -injective for $j \neq i$.*

Proof. This is a special case of [4, Theorem 13].

3. THE RESULT

Theorem. *Let R be a ring with ascending chain condition on essential left ideals.*

- (i) *If R is left or right continuous, then R is semiperfect.*
- (ii) *If R is left continuous, then R is left artinian.*
- (iii) *If R is right continuous, then R need not be left artinian.*

Proof. (i) By Lemma 2, $\bar{R} = R/J(R)$ is von Neumann regular and left (right) continuous. Because \bar{R} is regular, the right socle of \bar{R} = left socle of $\bar{R} = S$. Since \bar{R} also satisfies the acc on essential left ideals, it follows from Lemma

3 that $Q = \bar{R}/S$ is semisimple artinian. Let M be a singular left (right) \bar{R} -module. Since $SM = 0$ ($MS = 0$), M is a Q -module and therefore ${}_Q M(M_Q)$ is injective. This implies that M is injective as an \bar{R} -module because \bar{R} is regular. Next we show that every cyclic left (right) \bar{R} -module is continuous. For, let \bar{I} be a left (right) ideal of \bar{R} . Then \bar{I} is essential in \bar{A} , where $\bar{R} = \bar{A} \oplus \bar{B}$, since \bar{R} is continuous. Thus, \bar{A}/\bar{I} is singular and hence injective as an \bar{R} -module. Also, by Lemma 4, \bar{B} is \bar{A} -injective and so \bar{B} is \bar{A}/\bar{I} -injective. Now from $\bar{R}/\bar{I} \simeq \bar{A}/\bar{I} \times \bar{B}$, it follows by invoking again Lemma 4 in the other direction that \bar{R}/\bar{I} is continuous. Since a ring each of whose cyclic left (right) modules is continuous is always semiperfect ([3, p. 201]), the regular ring \bar{R} must be semisimple artinian. Hence by Lemma 2, R is semiperfect.

(ii) By Lemma 2 and Lemma 1, R is semiprimary. Write $R = \bigoplus_{i=1}^k \text{Re}_i$ as a direct sum of indecomposable left ideals. Since each Re_i is continuous, $\text{Soc}(\text{Re}_i)$ is simple or zero and so $\text{Soc}({}_R R)$ is finitely generated. Thus by Lemma 3, R is left noetherian and hence R is left artinian because R is semiprimary.

(iii) Let R be a ring with only three right ideals 0 , $J(R)$, and R , which is not left artinian (see [1, Example 7.11'.1, p. 337]). R is clearly right continuous. Also, R has acc on essential left ideals since $J(R) = \text{Soc}({}_R R)$ (Lemma 3). Incidentally, R is not even left noetherian.

This completes the proof.

4. REMARKS

(i) It follows from our theorem and that of Utumi, ([6, Theorem 7.10]) that a left and right continuous ring with acc on essential left and essential right ideals is quasi-Frobenius.

(ii) Contrary to the fact that for a right self-injective ring R the minimum condition on one side implies the minimum condition on the other side and the left self-injectivity of R , the example in part (iii) of the theorem shows that a right continuous right artinian ring need not be left artinian or left continuous.

(iii) A two-sided artinian ring with one-sided continuity is not necessarily continuous on the other side:

Let R be a ring with only three left ideals 0 , $J(R)$, and R , which is a right artinian with composition length 3 ([1, Example 7.11'.2, p. 338]). R is clearly left continuous. However, R is not right continuous. For otherwise, by Remark (i), R would be quasi-Frobenius, and hence uniserial with composition length 2 (on either side).

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DEPARTMENT OF MATHEMATICS, OHIO UNIVERSITY, ATHENS, OHIO 45701

(S. TARIQ RIZVI): DEPARTMENT OF MATHEMATICS, THE OHIO STATE UNIVERSITY, LIMA, OHIO 45804