

ON TOTALLY GEODESIC SPHERES IN GRASSMANNIANS AND $O(n)$

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ABSTRACT. It was shown in [5] that the generators of the homotopy groups of the stable orthogonal groups and the stable Grassmannians can be represented by embedded totally geodesic spheres of constant curvature. In this paper we prove that all elements of the above-mentioned homotopy groups can be represented by such spheres.

INTRODUCTION

The infinite orthogonal group O is, by definition, the direct limit of $O(n)$ with respect to the inclusion maps $O(n) \rightarrow O(n+1)$. Similarly, the infinite Grassmannian BO is defined as the direct limit of $G_n(\mathbf{R}^{2n})$ with respect to the inclusion maps $G_n(\mathbf{R}^{2n}) \rightarrow G_{n+1}(\mathbf{R}^{2n+2})$. BO is homotopy equivalent to the classifying space of O given by the construction of J. Milnor.

A map $f: S^m \rightarrow O$ (resp. $S^m \rightarrow BO$) is said to be *totally geodesic*, if it maps S^m totally geodesically into some $O(n)$ (resp. some $G_n(\mathbf{R}^{2n})$). In [5], Rigas proved that for every $m \geq 1$, the generators of the groups $\pi_m O$ and $\pi_m BO$ can be represented by totally geodesic spheres of constant curvature. In this paper, we will prove that exactly the same conclusion is true for *all* the elements of $\pi_m(O)$ and $\pi_m BO$. The totally geodesic spheres in $O(n)$ constructed in §1 are given by Clifford orthogonal multiplications while those in $G_n(\mathbf{R}^{2n})$ are given by orthogonal representations of $C_{0,m+1}$, the Clifford algebra on \mathbf{R}^{m+1} endowed with a *negative* definite inner product. These are, moreover, just the *isoclinic spheres* [7, 8] in the real Grassmannians. They also occur, in a different form, in the geometry of a class of isoparametric hypersurfaces in S^{2n-1} (cf. [2, 6]). The results in [6] enable us to identify them in $\pi_m BO$.

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Editor's Note: Professor Wang was tragically killed in an accident on April 7, 1989. He will be missed by his colleagues and friends.

Our approach, which makes use of results of Atiyah, Bott and Shapiro, is far simpler than that of [5].

1. CLIFFORD MODULES

We recall here a number of results connecting the Clifford algebra with the KO -groups of the spheres which are needed in the following sections. The reader is referred to [1] and [3] for details.

Let C_m be the Clifford algebra generated by $1, e_1, \dots, e_m$ subject to the relations

$$(1.1) \quad e_i e_j + e_j e_i = -2\delta_{ij}.$$

Since $C_{m-1} \cong C_m^0$ and $\text{Spin}(m)$ is a subgroup of the multiplicative group of invertible elements of C_m , a C_{m-1} module is canonically a $\text{Spin}(m)$ module.

$G = \text{Spin}(m+1)$ acts transitively on $S^m \subset C_{m+1}$ with isotropy subgroup $H = \text{Spin}(m)$. G is the total space of an H -principal bundle over S^m .

Let M be a C_{m-1} module; it is also an H -module. One constructs the vector bundle $\alpha(M) = G \times_H M$ associated to the H -principal bundle G . $\alpha(M)$ represents an element of $\widetilde{KO}(S^m)$, the group of stable bundles over S^m .

Every C_{m-1} module can be made orthogonal, i.e. one can put a positive definite inner product on M such that the multiplications by e_i are orthogonal transformations of M . Moreover, let \mathbf{R}^m be the vector subspace of C_{m-1} spanned by $1, e_1, \dots, e_{m-1}$ endowed with the inner product for which $1, e_1, \dots, e_{m-1}$ are orthonormal. Then it is easy to see that the Clifford multiplication

$$(1.2) \quad \mathbf{R}^m \times M \rightarrow M$$

is orthogonal, i.e. $\|\lambda x\| = \|\lambda\| \cdot \|x\|$. In this case, the vector bundle $\alpha(M)$ has a bundle metric hence a characteristic map $S^{m-1} \rightarrow O(l)$ where $l = \dim M$.

Lemma 1. *If M is an orthogonal C_{m-1} module, then the characteristic map $S^{m-1} \rightarrow O(l)$ is given by regarding S^{m-1} as the unit sphere of \mathbf{R}^m and composing with the orthogonal multiplication (1.2).*

Proof. This follows directly from Proposition (13.2) in [1].

It is well known from K -theory that $\pi_m BO \cong \widetilde{KO}(S^m) \cong \pi_{m-1} O$, the last isomorphism being given by the characteristic maps of vector bundles in $\widetilde{KO}(S^m)$.

Let $N(C_{m-1})$ be the free Abelian group generated by isomorphism classes of irreducible C_{m-1} modules. The map $M \mapsto \alpha(M)$ extends to a group homomorphism $\alpha: N(C_{m-1}) \rightarrow \widetilde{KO}(S^m)$. Since α clearly annihilates the image of $i^*: N(C_{m+1}) \rightarrow N(C_m)$, where i is the inclusion, it induces a homomorphism $\alpha: A_m \rightarrow \widetilde{KO}(S^m)$ where A_m is the cokernel of $i^*: N(C_m) \rightarrow N(C_{m-1})$.

The following is a special case of Theorem 11.5 in [1].

Lemma 2. α is an isomorphism of groups.

It is well known that when $m \not\equiv 0 \pmod{4}$, there is, up to isomorphism, only one irreducible C_{m-1} module. However, when $m \equiv 0 \pmod{4}$, there are two: Δ_m^+ and Δ_m^- (of the same dimension). Decompose M ; we can write $M = a\Delta_m^+ + b\Delta_m^-$ as C_{m-1} modules, where a and b are natural numbers. The number $q = a - b$ is called the index of M . It is the only algebraic invariant of the C_{m-1} module M . $\alpha(\Delta_m^+) + \alpha(\Delta_m^-) = 0$, since $\Delta_m^+ + \Delta_m^-$ is a C_m module.

Lemma 2 reduces the computation of $\widetilde{KO}(S^m)$ to a purely algebraic problem. For example the periodicity of $\pi_m O$ follows from that of A_m . But the specific results of the computations are not needed in this paper.

2. TOTALLY GEODESIC SPHERES IN $O(n)$

Let M be an orthogonal C_{m-1} module of dimension l . One has $l = k\delta(m)$, where k is a natural number, $\delta(m)$ is the dimension of irreducible C_{m-1} modules. Let $\mathbf{R}^m \times \mathbf{R}^l \rightarrow \mathbf{R}^l$ be the orthogonal multiplication defined by (1.2) and S^{m-1} be the unit sphere in \mathbf{R}^m . One has a natural map $\tilde{f}_M: S^{m-1} \rightarrow O(l)$ given by (1.2). Let $f_M = i \circ \tilde{f}_M$ where $i: O(l) \rightarrow O$ is the inclusion.

Theorem 1.

(a) f_M is a totally geodesic embedding of S^{m-1} onto a sphere of constant curvature in O .

(b) If $m \not\equiv 0 \pmod{4}$, $[f_M] = kg_m$; if $m \equiv 0 \pmod{4}$, $[f_M] = qg_m$, where g_m is a generator of $\pi_{m-1} O$ (g_m is 0 if $\pi_{m-1} O = 0$ by convention), q is the index of M defined in §1.

Proof. (b) follows directly from Lemma 1 and Lemma 2. (a) is a direct consequence of the following:

Lemma 3. Let V be a Euclidean vector space, $E \subset \text{End } V$ a vector subspace such that $E \cap O(V) = S(E)$, the unit sphere of E with respect to the inner product $\langle A, B \rangle = (1/\dim V) \text{tr } AB^t$. Then $S(E)$ is totally geodesic in $O(V)$.

Proof. Let γ be a geodesic in $S(E)$. $\gamma = S(E) \cap P$, where P is a 2-plane in E . We want to show that γ is a geodesic in $O(V)$.

Let T_1, T_2 be orthonormal in P . γ is the subset of E given by $\cos \theta T_1 + \sin \theta T_2$. Since γ lies on $O(V)$, a simple computation shows that $J = T_1' T_2$ is skew symmetric and $J^2 = -I$. Therefore

$$\cos \theta T_1 + \sin \theta T_2 = T_1(\cos \theta + \sin \theta J),$$

and the right-hand side is clearly a parametrization of a geodesic in $O(V)$.

Q.E.D.

Corollary 1. For all $m \geq 1$, every element of $\pi_m O$ can be represented by a totally geodesic sphere of constant curvature in O .

3. TOTALLY GEODESIC SPHERES IN $G_n(\mathbf{R}^{2n})$

Let $C_{0,m+1}$ be the Clifford algebra on \mathbf{R}^{m+1} with respect to a negative definite inner product on \mathbf{R}^{m+1} . An orthogonal representation of $C_{0,m+1}$ on \mathbf{R}^{2l} is uniquely determined by a set of $m+1$ elements $P_0, \dots, P_m \in O(2l)$ such that

$$(3.1) \quad P_i P_j + P_j P_i = 2\delta_{ij} I.$$

Let E be the $m+1$ -dimensional vector subspace of $\text{End}(\mathbf{R}^{2l})$ spanned by P_0, \dots, P_m and Σ be the unit m -sphere in E with respect to the inner product $\langle A, B \rangle = (1/2l) \text{tr} AB'$ on $\text{End}(\mathbf{R}^{2l})$. For any $P \in \Sigma$, it is clear that $P^2 = I$ and P is symmetric. The $+1$ and -1 eigenspaces of P are of dimension l .

There is a map $f_\Sigma: \Sigma \rightarrow G_l(\mathbf{R}^{2l})$ defined by

$$f_\Sigma(P) = +1 \text{ eigenspace of } P,$$

with the following property:

$$(3.2) \quad Q f_\Sigma(P) = f_\Sigma(QPQ), \quad P, Q \in \Sigma.$$

Proposition 1. f_Σ is a totally geodesic embedding onto a sphere of constant curvature in $G_l(\mathbf{R}^{2l})$.

Proof. The set (3.1) has the following property: for any $x \in \mathbf{R}^{2l}$ of unit length, the vectors $P_0 x, P_1 x, \dots, P_m x$ are orthonormal in \mathbf{R}^{2l} . Therefore, for $P, Q \in \Sigma$, $P \neq Q$ implies that $Px \neq Qx$. Hence $f_\Sigma(P) \cap f_\Sigma(Q) = \emptyset$ if $P \neq Q$. Since $\Sigma = E \cap O(\mathbf{R}^{2l})$, where E is $\text{Span}(P_0, \dots, P_m)$, it follows from Lemma 3 that Σ is totally geodesic in $O(2l)$.

Let F be the $+1$ eigenspace of P_0 . The action of $O(2l)$ on F gives a fibration $\pi_F: O(2l) \rightarrow G_l(\mathbf{R}^{2l})$. Then it is easy to verify that Σ is orthogonal to the fiber of π_F at P_0 . Since π_F is a Riemannian submersion, it follows that π_F maps geodesics of Σ emanating from P_0 to geodesics of $G_l(\mathbf{R}^{2l})$. Since P_0 is arbitrary and $\pi_F(\Sigma) = f_\Sigma(\Sigma)$, it follows that $f_\Sigma(\Sigma)$ is totally geodesic in $G_l(\mathbf{R}^{2l})$. Finally, let X, Y be 2 orthonormal tangent vectors to Σ at $P_0 \in O(2l)$ and K, K^* be the sectional curvature of $O(2l)$ and $G_l(\mathbf{R}^{2l})$, respectively. Since Σ is totally geodesic in $O(2l)$ and horizontal at P_0 , it follows immediately from O'Neill's curvature formula for submersions that $K^*(\pi_F(X), \pi_F(Y)) = K(X, Y) + 3\|A_X Y\|^2$ where $A_X Y = \frac{1}{2}[X, Y]^v$ is the fundamental tensor for π_F , cf. [4]. An easy computation shows that $\|A_X Y\| = 1$, thus $f_\Sigma(\Sigma)$ has constant curvature.

Remark 1. From [7] and [8] we know that the spheres $f_\Sigma(\Sigma)$ are *isoclinic* in that any two distinct l -planes in $f_\Sigma(\Sigma)$ are at constant angle to each other.

Remark 2. It follows from [7] that the converse of Proposition 1 is true. In other words, any isoclinic sphere in $G_l(\mathbf{R}^{2l})$ is of the form $f_\Sigma(\Sigma)$ for some P_0, \dots, P_m satisfying (3.1).

It remains to identify the homotopy classes of $i \circ f_\Sigma(\Sigma)$ in $\pi_m BO$, where $i: G_n(\mathbf{R}^{2n}) \rightarrow BO$ is the inclusion.

It is known that (cf., e.g., [2]) there is a 1-1 correspondence between orthogonal representations of C_{m-1} on \mathbf{R}^l and $C_{0,m+1}$ on \mathbf{R}^{2l} . In particular, one has $l = k\delta(m)$, k a natural number. Moreover

$$(3.3) \quad \text{tr}(P_0 \dots P_m) = 2q\delta(m).$$

Here q is called the index of P_0, \dots, P_m . If $m \not\equiv 0 \pmod{4}$, q is necessarily zero. If $m \equiv 0 \pmod{4}$, q can take any integer values between $-k$ and k subject only to the condition $q \equiv k \pmod{2}$.

Theorem 2. *If $m \not\equiv 0 \pmod{4}$, $[f_\Sigma(\Sigma)] = kg_m$. If $m \equiv 0 \pmod{4}$, $[f_\Sigma(\Sigma)] = qg_m$, where g_m is a generator of $\pi_m BO$ ($g_m = 0$ if $\pi_m BO = 0$ by convention).*

Proof. Let γ_l be the canonical rank l vector bundle over $G_l(\mathbf{R}^{2l})$ and $\xi = f_\Sigma^*(\gamma_l)$, the induced bundle over S^m . It was shown in [2] that the sphere bundle of ξ is diffeomorphic to the focal variety N of an isoparametric function on S^{2l-1} . In [6], Proposition 1 gave a bundle isomorphism

$$\alpha(M) \cong \xi,$$

where M is a C_{m-1} module of dimension l and index q . The theorem follows at once in view of Lemma 1 and Theorem 1.

Corollary. *For every $m \geq 1$, all elements of $\pi_m BO$ can be represented by totally geodesic spheres of constant curvature in BO .*

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REFERENCES

1. M. Atiyah, R. Bott and A. Shapiro, *Clifford modules*, Topology **3** (1964), suppl., 3–38.
2. D. Ferus, H. Karcher and H. Münzner, *Cliffordalgebren und neue isoparametrische Hyperflächen*, Math. Z. **177** (1981), 479–502.
3. D. Husemoller, *Fibre bundles*, McGraw-Hill, New York, 1966.
4. B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. **13** (1966), 459–469.
5. A. Rigas, *Geodesic spheres as generators of the homotopy groups of O and BO* , J. Differential Geom. **13** (1978), 527–545.
6. Q.-M. Wang, *On the topology of Clifford isoparametric hypersurfaces*, J. Differential Geom. **27** (1988), 55–66.
7. J. Wolf, *Geodesic spheres in Grassmann manifolds*, Illinois J. Math. **7** (1963), 425–446.
8. Y. C. Wong, *Isoclinic n -planes*, Mem. Amer. Math. Soc. **41** (1961).