PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 108, Number 3, March 1990

A PROBLEM IN ELECTRICAL PROSPECTION AND A n-DIMENSIONAL BORG-LEVINSON THEOREM

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(Communicated by Barbara L. Keyfitz)

Dedicated to M. Cotlar on his 75th birthday

ABSTRACT. We show that the Dirichlet to Neumann map for $-\Delta u+vu=0$, determines the potential $v(x)$, for $v(x)$ satisfying the condition of C. Fefferman and D. Phong.

We shall consider here a bounded domain $\Omega \subset \mathbf{R}^n$, $n \geq 3$ with smooth boundary. Consider then the equation in Ω given by $-\Delta u + vu = 0$. Define the Dirichlet to Neumann map Λ_v on $\partial\Omega$ given by

$$\Lambda_v(f) = \frac{\partial u}{\partial \nu}, \frac{\partial u}{\partial \nu} \text{ is the outward normal derivative}$$

and u solves the Dirichlet problem $-\Delta u + uv = 0$ in Ω and $u|_{\partial\Omega} = f$.

We recall, ([F], [CW]) the definition of the C. Fefferman, D. Phong class. We say $v \in F_p$ if for all cubes $Q \subset \mathbf{R}^n$,

$$\|v\|_{F_p} = \sup_Q |Q|^{2/n} \left[\frac{1}{|Q|}\int_Q |v|^p\right]^{1/p} < \infty.$$

We remark that $L^{n/2}(\mathbf{R}^n) \subset F_p$ for $p \leq n/2$, and likewise $L^{n/2,\infty} \subset F_p$, $p < n/2$. The containments are strict as $v = f(x/|x|)|x|^{-2}$, $f \in L^p(S^{n-1})$, $p > (n-1)/2$ is not in $L^{n/2,\infty}$ but $v \in F_p$, $p > (n-1)/2$. The main result proved here is as follows.

Theorem. *Suppose* $\|v_i\|_{F_p} \leq \varepsilon(n)$, $p > (n-1)/2$, $i = 1, 2$. *Assume that* $\Lambda_{v_1} = \Lambda_{v_2}$, *then* $v_1 = v_2$ *in* Ω.

Remark. In the theorem above it is enough to assume that v_i are supported in Ω.

The one-dimensional result is due to [B], [L]. If $v_i \in L^\infty$ the result is due to [NSU] and [HN]. The two-dimensional result is in [SU_1] and the C^∞ case in

Received by the editors February 8, 1989 and, in revised form, May 9, 1989.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 35P10, 42B20, 35B60, 35J10; Secondary 35J05, 35P20.

Supported in part by NSF grant DMS-8803493.

0002-9939/90 $1.00 + $.25 per page

[SU_2]. Applications to conductivity measurements are in [C], [KV], [SU_1] and [SU_2]. [KV] also treats the case where v_i are analytic.

The smallness assumption on the F_p norm is not needed if $v_i \in L^p(\Omega)$, $p > n/2$. We wish to thank D. Jerison and C. Kenig for pointing this out to us and also include their proof of this observation after the end of the proof in the main theorem above.

We recall the following inequality from [CS].

Theorem 1. *Let $f \in C_0^\infty(\mathbf{R}^n)$, $v > 0$ and $v \in F_p$, $p > (n-1)/2$. For $z \in \mathbf{C}^n$, $\gamma \in \mathbf{C}$, define $Q(D) = \Delta + z \cdot \nabla + \gamma$. Then*

$$\int_{\mathbf{R}^n} |f|^2 v \le c \int_{\mathbf{R}^n} |Q(D)f|^2 v^{-1}$$

where c is independent of f, z, γ.

We use the theorem stated above to prove the lemma that follows.

Lemma 1. *Let $v_i \in F_p \cap L^1$, $p > (n-1)/2$, and $\|v_i\|_{F_p} \le \varepsilon$, $i = 1, 2$. Let $z \in \mathbf{C}^n$ with $z \cdot z = 0$. Let $V(x) = |v_1| + |v_2| + \delta(1+|x|^2)^{-n}$, $\delta > 0$ and small. Let $L_V^2 = \left\{f\colon \int_{\mathbf{R}^n} |f|^2 V < \infty\right\}$. Then,*

(a) *there is a unique solution to $-\Delta + v_i$ of the form,*

$$u_i(x) = e^{z \cdot x} m_{z,i}(x), i = 1, 2 \text{ with } m_{z,i} \text{ in the space } L_V^2.$$

(b) *$\int_{\mathbf{R}^n} |m_{z,i}|^2 V \le c$, uniformly in z, $i = 1, 2$.*

(c) *$m_{z,i}(x) \to 1$, $i = 1, 2$ weakly in L_V^2 as $|z_k| \to \infty$, for some sequence z_k.*

Proof. Substituting $u = e^{z\cdot x} m_{z,i}(x)$ into $-\Delta u + v_i u = 0$, we note that $m_{z,i}(x)$ satisfies the equation

$$-\Delta m_{z,i} + (z \cdot \nabla) m_{z,i} + v_i m_{z,i} = 0.$$

Therefore, $m_{z,i}$ satisfies the integral equation,

$$m_{z,i} = 1 + G_z(v_i m_{z,i}), \qquad i = 1, 2 \tag{1}$$

where G_z denotes the Green function for $-\Delta + z \cdot \nabla$. Define $T_i f(x) = 1 + G_z(v_i f)(x)$. It will be enough for us to show T_i has a fixed point on the Banach space L_V^2, thus showing (a). In fact we show T_i is a contraction on L_V^2 and thus the uniqueness assertion (a) of Lemma 1 also follows. Since $v_1, v_2 \in L^1$, $V \in L^1$. Since v_1, v_2, $\delta(1+|x|^2)^{-n} \in F_p$, $p > (n-1)/2$, we have $\|V\|_{F_p} \le \varepsilon$, for $p > (n-1)/2$. Moreover $V > 0$. Thus, Theorem 1 is applicable with $v = V$. Moreover $|v_i| \le V$, $i = 1, 2$.

Thus,

$$\begin{aligned}\int_{\mathbf{R}^n} |T_i f|^2 V &\le \int_{\mathbf{R}^n} V + \int_{\mathbf{R}^n} |G_z(v_i f)|^2 V \\ &\le c + \varepsilon \int_{\mathbf{R}^n} |v_i f|^2 V^{-1} \le c + \varepsilon \int_{\mathbf{R}^n} |f|^2 V^2 V^{-1} \\ &\le \int_{\mathbf{R}^n} |f|^2 V < \infty .\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{\mathbf{R}^n} |T_i(f-g)|^2 V \le \int_{\mathbf{R}^n} |G_z(v_i(f-g)|^2 V &\le \varepsilon \int_{\mathbf{R}^n} |f-g|^2 |v_i|^2 V^{-1} \\ &\le \varepsilon \int_{\mathbf{R}^n} |f-g|^2 V .\end{aligned}$$

Thus T_i is a contraction and the existence and uniqueness of $m_{z,i}$ is assured. We now show (b). By (1),

$$\begin{aligned}\int_{\mathbf{R}^n} |m_{z,i}|^2 V &\le c \int_{\mathbf{R}^n} V + c \int_{\mathbf{R}^n} |G_z(v_i m_{z,i})|^2 V \\ &\le c_1 \int_{\mathbf{R}^n} V + \varepsilon c_2 \int_{\mathbf{R}^n} |m_{z,i}|^2 V\end{aligned}$$

where c_1, c_2 do not depend on z. Thus for small ε, and since from (a), $\int_{\mathbf{R}^n} |m_{z,i}|^2 V < \infty$, we see,

$$\int_{\mathbf{R}^n} |m_{z,i}|^2 V \le c, \qquad \text{uniformly in } z .$$

We now prove (c). Note the multiplier for G_z given by $(|\xi|^2 + iz\cdot\xi)^{-1} \to 0$ as $|z| \to \infty$. Next we note by (b),

$$\int_{\mathbf{R}^n} |G_z(v_i m_{z,i})|^2 V \le \varepsilon \int_{\mathbf{R}^n} |m_{z,i}|^2 V \le c .$$

Thus there is a sequence z_k, $|z_k| \to \infty$, so that $G_{z_k}(v_i m_{z_k,i}) \rightharpoonup 0$ weakly in L^2_V. Since (1) holds, it follows that $m_{z_k,i} \rightharpoonup 1$ as $|z_k| \to \infty$ in L^2_V. The lemma is now proved.

Lemma 2. *Extend* v_1 *and* v_2 *to be zero outside* Ω. *Let* u_i *be the unique solutions of Lemma* 1 *to* $-\Delta + v_i$, $i = 1, 2$. *If* $\Lambda_{v_1} = \Lambda_{v_2}$, *then* $u_1 = u_2$ *in* $\mathbf{R}^n \setminus \Omega$.

Proof. Recall, [F], [CW], that if $\|v_i\|_{F_p} \le \varepsilon$, $p > 1$, then for $f \in C_0^\infty(\mathbf{R}^n)$,

$$\int_{\mathbf{R}^n} |f|^2 |v_i| \le \varepsilon \int_{\mathbf{R}^n} |\nabla f|^2 . \tag{2}$$

Thus the bilinear form,

$$\int_\Omega \nabla u \cdot \nabla \varphi + \int_\Omega u \varphi v_2$$

is coercive and continuous for u, $\varphi \in H_0^1(\Omega)$. As,

$$\int_\Omega |m_{z,1}|^2|v_2| \le \int_{\mathbf{R}^n} |m_{z,1}|^2|v_2| \le \int_{\mathbf{R}^n} |m_{z,1}|^2 V < \infty,$$

and $u_1 = e^{z\cdot x} m_{z,1}$, we get $\int_\Omega |u_1|^2|v_2| < \infty$. Thus, the Dirichlet problem,

$$\begin{aligned} -\Delta u + uv_2 &= 0 \qquad \text{in } \Omega \\ u &= u_1 \qquad \text{on } \partial\Omega \end{aligned}$$

has a unique solution u, such that $\int_\Omega |u|^2|v_2| < \infty$.

Since $u - u_1$ has compact support, by (2), as $V \in F_p$, $p > 1$,

$$\int_\Omega |u - u_1|^2 V \le \int_\Omega |\nabla(u - u_1)|^2 < \infty.$$

On Ω, $|u_1| \le c|m_{z,1}|$, thus, $\int_\Omega |u|^2 V < \infty$ by Lemma 1.

Define,

$$\Phi = \begin{cases} u & \text{in } \Omega \\ u_1 & \text{in } \mathbf{R}^n \setminus \Omega. \end{cases}$$

Since $\Lambda_{v_2}(u_1) = \partial u/\partial\nu = \Lambda_{v_1}(u_1) = \partial u_1/\partial\nu$, Φ is a solution to $\mathbf{R}^n$ to $-\Delta + v_2$. Writing $\Phi = e^{z\cdot x}[e^{-z\cdot x}\Phi] = e^{z\cdot x} M_z(x)$, we see that $M_z(x) = m_{z,1}(x)$ in $\mathbf{R}^n \setminus \Omega$, and since $\int_\Omega |u|^2 V < \infty$, it follows that $\int_{\mathbf{R}^n} |M_z|^2 V < \infty$. By the uniqueness assertion of Lemma 1, $M_z = m_{z,2}$, and thus $\Phi = u_2$, in particular $u_1 = u_2$ in $\mathbf{R}^n \setminus \Omega$.

We are now in a position to prove our main theorem.

Proof. Fix $l \in \mathbf{Z}^n$. Choose k, $e \in \mathbf{R}^n$, so that $|k| = |l - e|$, $k \cdot e = k \cdot l = e \cdot l = 0$. This choice forces $|k| = |l + e|$. Let $z = \frac{1}{2}(-k + i(l - e))$, $\tilde{z} = \frac{1}{2}(k + i(l + e))$. Note $z \cdot z = \tilde{z} \cdot \tilde{z} = 0$. We shall use Green's theorem in the form,

$$\int_\Omega (w\Delta f - f\Delta w) = \int_{\partial\Omega} \left[w\frac{\partial f}{\partial\nu} - f\frac{\partial w}{\partial\nu} \right] d\sigma$$

with the choice $w = e^{\tilde{z}\cdot x}$, $f = u_i = e^{z\cdot x} m_{z,i}$. Let D_ρ be a collar neighborhood of $\partial\Omega$ with thickness ρ. Let $\partial D_\rho \cap \Omega = \partial D_{\rho,1}$ and $\partial D_\rho \cap (\mathbf{R}^n \setminus \Omega) = \partial D_{\rho,2}$. Note

$$\int_\Omega e^{\tilde{z}\cdot x} v_i u_i = -\int_\Omega e^{\tilde{z}\cdot x} \Delta u_i = \int_{\partial\Omega} e^{\tilde{z}\cdot x} \left[(\tilde{z} \cdot \nu) u_i - \frac{\partial u_i}{\partial\nu} \right] d\sigma. \tag{3}$$

We now show that,

$$\int_{\partial\Omega} e^{\tilde{z}\cdot x} \left[(\tilde{z} \cdot \nu) u_1 - \frac{\partial u_1}{\partial\nu} \right] d\sigma = \int_{\partial\Omega} e^{\tilde{z}\cdot x} \left[(\tilde{z} \cdot \nu) u_2 - \frac{\partial u_2}{\partial\nu} \right] d\sigma. \tag{4}$$

Temporarily assume (4). Combining (3) and (4) we see

$$\int_\Omega e^{\tilde{z}\cdot x} v_1 u_1 = \int_\Omega e^{\tilde{z}\cdot x} v_2 u_2.$$

Since $u_i = e^{z\cdot x} m_{z,i}$, we get

$$\int_\Omega e^{il\cdot x} m_{z,1} v_1 = \int_\Omega e^{il\cdot x} m_{z,2} v_2 .$$

Letting $|z| \to \infty$ and using (c) of Lemma 1, we conclude

$$\int_\Omega e^{il\cdot x} v_1 = \int_\Omega e^{il\cdot x} v_2 .$$

This shows $v_1 = v_2$. We now show (4). We apply Green's theorem to the collar neighborhood D_ρ. Since $v_1 = v_2 = 0$ in $\mathbf{R}^n \setminus \Omega$, u_1 and u_2 are harmonic and C^∞ in $\mathbf{R}^n \setminus \Omega$. By Lemma 2, $u_1 = u_2$ in $\mathbf{R}^n \setminus \Omega$. Thus

$$\int_{\partial D_{\rho,2}} e^{\tilde{z}\cdot x} \left[(\tilde{z}\cdot\nu) u_1 - \frac{\partial u_1}{\partial \nu} \right] d\sigma = \int_{\partial D_{\rho,2}} e^{\tilde{z}\cdot x} \left[(\tilde{z}\cdot\nu) u_2 - \frac{\partial u_2}{\partial \nu} \right] d\sigma .$$

So by Green's theorem

$$\int_{\partial D_{\rho,1}} e^{\tilde{z}\cdot x} \left[(\tilde{z}\cdot\nu)(u_1 - u_2) - \frac{\partial u_1}{\partial \nu} + \frac{\partial u_2}{\partial \nu} \right] d\sigma = \int_{D_\rho} e^{\tilde{z}\cdot x} (v_1 u_1 - v_2 u_2) .$$

But $\int_{D_\rho} |u_i|^2 V < \infty$. Thus as $\rho \to 0$, the integral on the right side converges to zero. Thus the integral on the left side converges to zero. But in the limit the integral on the left side is exactly the difference of the two integrals in (4). This establishes (4).

We now give the argument by D. Jerison and C. Kenig. In essence we show that a form of Lemma 1 holds with no smallness assumption if $v_i \in L^r(\Omega)$, $r > n/2$.

We begin with,

Lemma 3. *Let $2/(n+1) \le (q-2)/q \le 2/n$. Let $|z| = 1$, and $z \cdot z = 0$. Then for $1/p + 1/q = 1$,*

$$\|G_z f\|_{L^q(\mathbf{R}^n)} \le c \|f\|_{L^p(\mathbf{R}^n)}$$

where $\widehat{G_z f}(\xi) = (|\xi|^2 + z\cdot\xi)^{-1} \hat{f}(\xi)$ and c is independent of f and z.

Proof. We assume w.l.o.g. that

$$\widehat{G_z f}(\xi) = (|\xi|^2 - 2\xi_1 + 2i\xi_2)^{-1} \hat{f}(\xi) .$$

By changing variables in ξ_1, $\xi_1 \to (\xi_1 - 1)$ we can assume that $\widehat{G_z f}(\xi) = (|\xi|^2 + 1 + 2i\xi_2)^{-1} \hat{f}(\xi)$. Since $1/p + 1/q = 1$, $(q-2)/q = 1/p - 1/q$, and thus under the hypothesis of the lemma, $2/(n+1) \le 1/p - 1/q \le 2/n$. We may thus apply Theorem 2.4 in [KRS] to conclude Lemma 3.

From Lemma 3 we deduce the next lemma. The notation we adopt is identical to Lemma 1.

Lemma 4. *Let $w_{z,i}(x) = m_{z,i}(x) - 1$. Let $v_i \in L^r(\Omega)$, $r > n/2$, and $z \cdot z = 0$. Let $2/(n+1) \le (q-2)/q = 1/r < 2/n$. Then, for $|z|$ large,*

(a) *there is a unique solution to $-\Delta + v_i$ of the form $u_i(x) = e^{z\cdot x} m_{z,i}(x)$, with $\|w_{z,i}\|_{L^q(\mathbf{R}^n)} \le c$ uniformly in z.*

(b) $\|w_{z,i}\|_{L^q(\mathbf{R}^n)} \to 0$ *as* $|z| \to \infty$.

Proof. From (1) we readily see that $w_{z,i}$ satisfies

$$w_{z,i} + G_z(v_i w_{z,i}) = G_z(v_i \chi_\Omega).$$

Let $M_{v_i}(f) = v_i f$, the multiplication by v_i operator. The identity above can be rewritten as,

$$(I + G_z M_{v_i})(w_{z,i}) = G_z M_{v_i}(\chi_\Omega), \tag{5}$$

where $I =$ identity operator. We now claim that for $\alpha = 2 - n/r > 0$, and for c independent of z,

$$\|G_z M_{v_i}(f)\|_{L^q(\mathbf{R}^n)} \le c|z|^{-\alpha}\|f\|_{L^q(\mathbf{R}^n)}. \tag{6}$$

Temporarily assume (6) and note that for large $|z|$, $I + G_z M_{v_i}$ is invertible on $L^q(\mathbf{R}^n)$, and

$$\|G_z M_{v_i}(\chi_\Omega)\|_{L^q(\mathbf{R}^n)} \le c|z|^{-\alpha}|\Omega|^{1/q},$$

c independent of z. Thus the uniqueness and existence of $w_{z,i}$ follows from (5) and $\|w_{z,i}\|_{L^q(\mathbf{R}^n)} \le c|z|^{-\alpha}$. So we are reduced to checking (6). Let $\delta = |z|$, and $T_\delta f(x) = f(\delta x)$. We note $G_z = \delta^{-2} T_\delta G_{z\delta^{-1}} T_{\delta^{-1}}$ by a change of variables, and moreover $\|T_\delta f\|_{L^s(\mathbf{R}^n)} = \delta^{-n/s}\|f\|_{L^s(\mathbf{R}^n)}$. Thus,

$$\|G_z M_{v_i}(f)\|_{L^q} = \delta^{-2-n/q}\|G_{z\delta^{-1}} T_{\delta^{-1}} M_{v_i} f\|_{L^q}.$$

By Lemma 3, the right side above is at most

$$c\delta^{-2-n/q}\|T_{\delta^{-1}} M_{v_i}(f)\|_{L^p} \le c\delta^{-2-n/q+n/p}\|v_i f\|_{L^p}, \qquad 1/p + 1/q = 1.$$

Now $r^{-1} = 1 - 2q^{-1} = p^{-1} - q^{-1}$, because $p^{-1} + q^{-1} = 1$. So the right side above is at most $c\delta^{-2+n/r}\|v_i f\|_{L^p}$. Now applying Holder's inequality with exponents r/p and $r/(r-p) = q/p$,

$$\begin{aligned} c\delta^{-2+n/r}\|v_i f\|_{L^p} &\le c\delta^{-2+n/r}\|v_i\|_{L^r(\Omega)}\|f\|_{L^q(\mathbf{R}^n)} \\ &\le |z|^{-\alpha}\|f\|_{L^q(\mathbf{R}^n)}. \end{aligned}$$

Thus we have (6), and Lemma 4 follows.

Using Lemma 4 we may conclude the fact that Λ_v determines v exactly as before.

References

[BC] R. Beals and R. Coifman, *Multidimensional inverse scatterings and non-linear partial differential equations*, Proc. Sympos. Pure Math. **43** (1984), 45–70.

[B] G. Borg, *Eine Umkerung der Sturm-Liouville Eigenwertarfgabe. Bestimmung def Differentialgleichung durch die Eigenwerte*, Acta Math. 78 (1946), 1–96.

[C] A. P. Calderón, *On an inverse boundary value problem*, Seminar on numerical analysis and its applications to continuum physics, Soc. Brasiliera de Mathematica, Rio de Janeiro (1980), 65–73.

[CS] S. Chanillo and E. Sawyer, *Unique continuation for $\Delta + v$ and the C. Fefferman-Phong class*, Trans. Amer. Math. Soc. (to appear).

[CW] S. Chanillo and R. Wheeden, *L^p estimates for fractional integrals and Sobolev inequalities with applications to Schrödinger operators*, Comm. Partial Differential Equations **10** (1985), 1077–1116.

[F] C. Fefferman, *The uncertainty principle*, Bull. Amer. Math. Soc. **9** (1983), 129–206.

[GL] I. M. Gelfand and B. M. Levitan, *On the determination of a differential equation from its spectral function*, Izv. Akad. Nauk. SSSR Ser. Math. **15** (1961), 309–360.

[HN] G. M. Henkin and R. G. Novikov, Uspekhi Mat. Nauk. **42** (1987), 94–153.

[KRS] C. E. Kenig, A. Ruiz and C. Sogge, *Sobolev inequalities and unique continuation for second order constant coefficient differential equations*, Duke Math. J. **55** (1987), 329–348.

[KV] R. Kohn and M. Vogelius, *Determining conductivity by boundary measurements*, Comm. Pure Appl. Math. **37** (1984), 289–298.

[L] N. Levinson, *The inverse Sturm-Liouville problem*, Math. Tidsskr. B. (1949), 25–30.

[NSU] A. Nachman, J. Sylvester and G. Uhlmann, *An n-dimensional Borg-Levinson theorem*, Comm. Phys. Math. (to appear).

[SU_1] J. Sylvester and G. Uhlmann, *A uniqueness theorem for an inverse boundary value problem in electrical prospection*, Comm. Pure Appl. Math. **39** (1986), 91–112.

[SU_2] ——, *A global uniqueness theorem for an inverse boundary value problem*, Ann. of Math. **125** (1987), 153–169.

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