

ON COMPOSITIONS OF DERIVATIONS OF PRIME RINGS

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ABSTRACT. Let R be a prime ring and $\varphi(x_i)$ be a differential polynomial of R . It is shown that if $\varphi(x_i) = 0$ holds on a nonzero two-sided ideal of R , then $\varphi(x_i) = 0$ holds on R_F , the left Martindale quotient ring of R . Using this together with Kharchenko's theorem on differential identities, we settle three problems raised by Krempa and Matczuk in the positive.

1. INTRODUCTION

In [4], Krempa and Matczuk raised the following problems:

Problem 1. Let L be a field, $\text{char } L = p < \infty$, and let $\delta_1, \dots, \delta_p$ be such derivations of L that $\varphi(x) = x^{\delta_1, \dots, \delta_p}$ defines a derivation. Whether there exists $l \in L$ such that $\varphi(x) = lx^{(d^p)}$ for some derivation d of L .

In the following, let $\delta_1, \dots, \delta_n$ be derivations of a prime ring, $\text{char } R > n$, and let I be a nonzero ideal of R .

Problem 2. Whether $x^{\delta_1, \dots, \delta_n} = cx$ for all $x \in I$, where c is a fixed element in the extended centroid of R , implies $c = 0$.

Problem 3. Whether $x^{\delta_1, \dots, \delta_n} = 0$ for all $x \in I$ implies $x^{\delta_1, \dots, \delta_n} = 0$ for all $x \in R$.

Our aim here is to show that all the three problems follow easily from Kharchenko's powerful theorem on differential identities.

2. PRELIMINARY

Here we recall Kharchenko's theorem on differential identities and meanwhile explain our notation (adopted from [2] and [3]).

Throughout the paper, R is a prime ring and F is the filter of all nonzero two-sided ideals of R . Let R_F and Q stand for, respectively, the left Martindale quotient ring of R . (See p. 156 of [2] for the definitions.) The center of R_F , denoted by C , coincides with the center of Q and is called the extended

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centroid of R . A derivation δ of the ring R is a map from R into R satisfying $(x + y)^\delta = x^\delta + y^\delta$ and $(xy)^\delta = x^\delta y + xy^\delta$ for all $x, y \in R$. Any derivation δ defined on R can be *uniquely* extended to a derivation of R_F , which will also be denoted by δ . Thus all derivations of R here are implicitly assumed to be defined on the whole R_F . For a derivation δ of R_F and for $\alpha \in C$, we define $\delta\alpha$ by $x^{\delta\alpha} = \alpha x^\delta$ for all $x \in R_F$. So $\delta\alpha$ is also a derivation of R_F . In this manner, the set of all derivations of R_F forms a right C -vector space.

A derivation δ is said to be (Q) -inner if there exists $a \in Q$ such that $x^\delta = [x, a] = xa - ax$ for all $x \in R_F$.

Let D_{int} be the set of all inner derivations of R_F . D_{int} is obviously a C -subspace of the space of all derivations of R_F . Let $\text{Der}(R)$ be the set of all derivations of R and let D be the C -subspace spanned by $\text{Der}(R)$, that is, $D = \text{Der}(R) \cdot C$. The C -subspace spanned by all derivations of R and all (Q) -inner derivations is obviously equal to $D_{\text{int}} + D$.

We choose a basis M_0 for D_{int} and augment it to a basis M of $D_{\text{int}} + D$. We also fix a total order $>$ in the set M such that $\mu_0 > \mu$ for $\mu_0 \in M_0$ and $\mu \in M \setminus M_0$. We then extend this order to the set of all derivation words in M by assuming that a longer word is greater than a shorter one and that words of the same length are ordered lexicographically. By a *regular word* in M , we mean a word of the form $\Delta = \delta_1^{s_1} \delta_2^{s_2} \cdots \delta_m^{s_m}$ such that (1) $\delta_i \in M \setminus M_0$ for $i = 1, \dots, m$, (2) $\delta_1 < \delta_2 < \cdots < \delta_m$, and (3) $s_i < p$ for $i = 1, \dots, m$, if $\text{char } R = p > 0$.

Remark 1. Since $\text{Der}(R)$ spans D as a C -subspace, elements of $M \setminus M_0$ can be chosen from $\text{Der}(R)$. In the proof of the crucial Lemma 2 of [2] (pp. 158–160), derivations in $M \setminus M_0$ are implicitly assumed to be in $\text{Der}(R)$ and this can be done as is said above. However, as is explained in the remark on p. 74 of [3], all we need is that for each derivation $\delta \in M \setminus M_0$, there exists a nonzero ideal I of R such that $I^\delta \subseteq R$. Every (Q) -inner derivation satisfies this property by the very definition of Q , and every derivation of D also satisfies this property, since so does every derivation in $\text{Der}(R)$. So this property is obviously enjoyed by all derivations of $D_{\text{int}} + D$. For this reason, we will *not* put any restriction on M .

By a differential polynomial, we mean a generalized polynomial with coefficients in R_F and involving noncommuting indeterminates which are acted by compositions of *derivations of* R . For a differential polynomial φ , $\varphi = 0$ is said to be a differential identity for R if φ assumes the constant value 0 for any assignment of values from R to its indeterminates.

By means of identities 1–6, [3, p. 58], any differential polynomial $\varphi(x_i)$ can be transformed into the form $\psi(x_i^{\Delta_j})$, where (1) $\psi(z_{ij})$ is a generalized polynomial in distinct indeterminates z_{ij} , and where (2) Δ_j are *regular words* in

M . $\psi(x_i^{\Delta_j})$ is called the reduced expression of $\varphi(x_i)$. Kharchenko's theorem on differential identities [3, p. 65] asserts that if $\psi(x_i^{\Delta_j}) = 0$ is a differential identity of R , where Δ_j are distinct regular derivation words in M and where $\psi(z_{ij})$ is a generalized polynomial, then $\psi(z_{ij}) = 0$ is a generalized polynomial identity of R . The remark on page 74 of [3] says that the same conclusion still holds if $\varphi(x_i) = 0$ is only assumed to hold on a nonzero ideal of R instead of the whole R .

Remark 2. In Kharchenko's theorem quoted above, as long as two regular derivation words Δ, Δ' in M are distinct as words, x^Δ and $x^{\Delta'}$ must be substituted by distinct indeterminates, no matter whether Δ, Δ' are the same as maps on R or not, that is, no matter whether $x^\Delta = x^{\Delta'}$ holds for all $x \in R$ or not. As a consequence, two distinct regular derivation words Δ, Δ' can never be the same as maps on R (or on a nonzero ideal of R): For if $x^\Delta = x^{\Delta'}$ holds for all $x \in R$, then, by Kharchenko's theorem, the identity $y = z$, where y, z are distinct indeterminates, would hold on R , an obvious contradiction. As an immediate corollary, if $\delta \in M \setminus M_0$, then δ^t ($1 < t < \text{char } R$) can never be equal to any $\mu \in M \setminus M_0$ as maps, since, as regular words, δ^t and μ are of length t ($t > 1$) and of length one, respectively, and hence must be distinct as words. Actually, if $\delta \in M \setminus M_0$, then δ^t ($1 < t < p$) cannot be a derivation in $D_{\text{int}} + D$ (as maps): Indeed, if δ^t is a derivation in $D_{\text{int}} + D$, then δ^t must be a C -linear combination of derivations in M (as maps). So there exist $\alpha_i \in C$, $\mu_i \in M \setminus M_0$ and $b \in Q$ such that $x^{\delta^t} = \sum_i \alpha_i x^{\mu_i} + [b, x]$ holds for all $x \in R$. By Kharchenko's theorem, the identity $y = \sum \alpha_i z_i + [b, x]$, where y, z_i, x are distinct indeterminates, holds on R . This is absurd. These comments will help clarify parts of the argument (Cases 1 and 2) in the proof of Theorem 2.

3. MAIN RESULTS

The following theorem is simply a combination of a remark in [3] and a theorem in [1], and can be viewed as a generalization of both.

Theorem 1. Let R be a prime ring and let R_F be its left Martindale quotient ring. Assume that $\varphi(x_i)$ is a differential polynomial involving only derivations of R , in noncommuting indeterminates x_i and with coefficients in R_F . If $\varphi(x_i)$ vanishes on a nonzero ideal of R , then $\varphi(x_i)$ vanishes on R_F .

Proof. Using the basic identities 1–6 [3, p. 58] $\varphi(x_i)$ can be transformed into the reduced expression $\psi(x_i^{\Delta_j})$, where Δ_j are regular derivation words and where $\psi(z_{ij})$ is an ordinary generalized polynomial. By the remark on page 74 of [3], $\psi(z_{ij}) = 0$ is a generalized polynomial identity of R . By Theorem 2 [1], $\psi(z_{ij}) = 0$ is also a generalized polynomial identity of R_F . Thus $\psi(x_i^{\Delta_j}) = 0$ also holds on R_F , when the derivations are extended to be defined on R_F . Since the basic identities 1–6 [3, p. 58] also hold for the

derivations thus extended, $\psi(x_i^{\Delta_j}) = 0$ can be transformed back to $\varphi(x_i) = 0$. So $\varphi(x_i) = 0$ holds on R as desired.

Actually, Theorem 1 is more or less implicitly implied in [3]. If we refer to the proof of [3] instead of theorems only, a more direct proof of Theorem 1 can be given.

Obviously, Problem 3 is only a very special instance of the above theorem. Now we prove Problem 2 (without the assumption that $\text{char } R > n$).

Proof of Problem 2. Suppose that $\delta_1, \dots, \delta_n$ are derivations of a prime ring R such that $x^{\delta_1 \cdots \delta_n} = cx$ for all $x \in I$, where I is a nonzero ideal of R and c is a fixed element of C . The differential identity $x^{\delta_1 \cdots \delta_n} = cx$ holds on I and hence must also hold on R_F by Theorem 1. Since R_F has the identity 1, we have $c = c1 = (1)^{\delta_1 \cdots \delta_n} = 0$, as desired.

Remark 3. (1) The above proof also holds for a C -linear combination of compositions of derivations. For example, let $\delta_1, \dots, \delta_n, d_1, \dots, d_m, \mu_1, \dots, \mu_l, \dots$ be derivations of R and let $\alpha, \beta, \gamma, \dots \in C$. If $\alpha x^{\delta_1 \cdots \delta_n} + \beta x^{d_1 \cdots d_m} + \gamma x^{\mu_1 \cdots \mu_l} + \dots = cx$ for all $x \in I$ and for some fixed $c \in C$, then $c = 0$. (2) In the assumption, c can also be assumed to be dependent on x , that is, $c = c(x)$. An easy argument shows that $c(x)$ is a constant, unless R is commutative.

For $a_1, \dots, a_m \in R$, let $[x, a_1, \dots, a_m]$ denote $[[[x, a_1], a_2], \dots], a_m]$. The proof of the following theorem, though a little bit lengthy, is still a straight forward application of Kharchenko's theorem.

Theorem 2. Let R be a prime ring of characteristic $p > 0$. Suppose that $\delta_1, \dots, \delta_p$ are derivations of R such that $\varphi(x) = x^{\delta_1 \cdots \delta_p}$ defines a derivation on R . Then (1) if one of $\delta_1, \dots, \delta_p$ is inner, then $\varphi(x)$ is also inner; (2) if all $\delta_1, \dots, \delta_p$ are outer, then there exists $0 \neq \alpha \in C$ such that $\varphi(x) = \alpha x^{(\delta_1^p)}$.

Proof. Let δ be any derivation on R and let $a_1, \dots, a_m \in R_F$ ($m \geq 1$). Observe that the identity

$$[x, a_1, \dots, a_m]^\delta = [x^\delta, a_1, \dots, a_m] + [x, a_1^\delta, \dots, a_m] + \dots + [x, a_1, \dots, a_m^\delta]$$

holds for all $x \in R$. Using this identity together with the basic identities 1–6 [3, p. 58], the reduced expression $\psi(x)$ of $\varphi(x) = x^{\delta_1 \cdots \delta_p}$ can be written as a sum of the terms of the form αx^Δ or $[x^\Delta, a_1, \dots, a_m]$ ($m \geq 1$), where Δ is a regular derivation word, $\alpha \in C$ and $a_1, \dots, a_m \in R_F$.

Since $\varphi(x)$ defines a derivation on R , there exist $\mu_j \in M \setminus M_0$, $\beta_j \in C$ and $b \in Q$ such that $\varphi(x) = \sum_j \beta_j x^{\mu_j} + [x, b]$. Hence

$$\psi(x) = \sum_j \beta_j x^{\mu_j} + [x, b]$$

gives a reduced linear differential identity, to which we are going to apply Theorem 2 [3, p. 65].

First, assume that one of δ_i ($i = 1, \dots, p$) is inner. Then $\psi(x)$ involves only terms of the form $[x^\Delta, a_1, \dots, a_m]$ ($m \geq 1$), where Δ are regular derivation words. Let

$$\sum_s [x^{\mu_j}, a_1^{(s)}, a_2^{(s)}, \dots, a_{m(s)}^{(s)}], \quad (m(s) \geq 1)$$

denote the sum of all terms in $\psi(x)$ which involve the single derivation word μ_j . (Set $s = 0$, if no such terms exist.) By Theorem 2 [3],

$$\sum_s [z, a_1^{(s)}, \dots, a_{m(s)}^{(s)}] = \beta_j z$$

is a GPI of R and hence a GPI of R_F by Theorem 1 above or Theorem 2 [1]. Set $z = 1$. Then $\beta_j = \beta_j 1 = \sum_s [1, a_1^{(s)}, \dots, a_{m(s)}^{(s)}] = 0$ ($m(s) \geq 1$). So $\beta_j = 0$ for each μ_j . Thus $\varphi(x) = [x, b]$ is inner as desired.

So, from now on, we assume that all δ_i ($i = 1, \dots, p$) are outer. Without loss of generality, we may assume that δ_1 is the minimal element in M with respect to $<$.

Suppose that one of $\delta_2, \dots, \delta_p$ is C -independent of δ_1 modulo D_{int} . Let $s > 1$ be the minimal integer $\leq p$ such that δ_s is independent of δ_1 modulo D_{int} . Without loss of generality, we may also assume that δ_s is the next minimal element of M . Then the leading word Δ of $\psi(x)$ must be of the form $\Delta = \delta_1^{t_1} \delta_s^{t_2} \dots (t_1 \geq 1, t_2 \geq 1)$. Since the characteristic of R is p and since all δ_i are outer, the only term of $\psi(x)$ which involves x^Δ is of the form αx^Δ , where α is a nonzero element of the extended centroid C . Theorem 2 [3] says that $\alpha z = 0$ is a GPI of R . This is obviously a contradiction.

So we must assume that each of $\delta_2, \dots, \delta_p$ is C -dependent on δ_1 modulo D_{int} . For brevity, set $\delta = \delta_1$. Using the basic identities 1–6 [3, p. 58], $\varphi(x)$ can be written in the form $\varphi(x) = \alpha x^{(\delta^p)} + \varphi'(x)$, where $0 \neq \alpha \in C$ and where $\varphi'(x)$ involves only regular derivation words $\delta, \delta^2, \dots, \delta^{p-1}$. By means of the identity mentioned at the beginning of the proof, $\varphi'(x)$ can be written as a sum of terms of the form $[x^{(\delta^t)}, a_1, \dots, a_m]$, ($1 \leq t \leq p-1, m \geq 1$). It is important to observe that $\varphi'(x)$ contains no terms involving x .

Case 1. δ and δ^p are C -dependent modulo D_{int} : Let us say $x^{\delta^p} = \gamma x^\delta + [x, c]$, where $\gamma \in C$ and $c \in Q$. The following facts are obvious: (1) the regular derivation words occurring in ψ are \emptyset (empty), $\delta, \delta^2, \dots, \delta^{p-1}$; (2) $\alpha[x, c]$ is the only term of $\psi(x)$ which involves x .

Now, we apply Theorem 2 [3]: By (2), $\alpha[x, c] = [x, b]$ for all $x \in R$. By (1), $\beta_j = 0$ for each $\mu_j \neq \delta$. (Note that δ^t ($1 < t < p$) is distinct from μ_j as regular derivation words. See Remark 2.) For $\mu_j = \delta$, let $\sum_s [x^\delta, a_1^{(s)}, \dots, a_{m(s)}^{(s)}]$ be the sum of all those terms in $\varphi'(x)$ which involve x^δ . By Theorem 2 [3],

$$\gamma z + \sum_s [z, a_1^{(s)}, \dots, a_{m(s)}^{(s)}] = \beta_j z$$

is a GPI of R and hence a GPI of R_F by Theorem 1. Hence

$$\beta_j = \beta_j 1 = \alpha \gamma 1 + \sum_s [1, a_1^{(s)}, \dots, a_{m(s)}^{(s)}] = \alpha \gamma.$$

So $\varphi(x) = \beta_j x^\delta + [x, b] = \alpha \gamma x^\delta + \alpha [x, c] = \alpha x^{\delta^p}$, as desired.

Case 2. δ and δ^p are C -independent modulo D_{int} : Without loss of generality, we may also assume that $\delta^p \in M$. Hence δ^p , regarded as a regular word, is of length one. Thus the only regular derivation words occurring in $\psi(x)$ are δ , $\delta^2, \dots, \delta^p$ (and no empty word \emptyset).

Now, we apply Theorem 2 [3]: For $\mu_j \neq \delta, \delta^p$, (as derivation words), we have $\beta_j = 0$ as before. (See Remark 2.) For $\mu_j = \delta$, let $\sum_s [x^\delta, a_1^{(s)}, \dots, a_{m(s)}^{(s)}]$ be the sum of those terms in $\varphi'(x)$ which involve x^δ . Then, by Theorem 2 [3],

$$\sum_s [z, a_1^{(s)}, \dots, a_{m(s)}^{(s)}] = \beta_j z$$

is a GPI of R and hence a GPI of R_F . So $\beta_j = \beta_j 1 = \sum_s [1, a_1^{(s)}, \dots, a_{m(s)}^{(s)}] = 0$. For $\mu_j = \delta^p$, $\beta_j = \alpha$ follows easily. Since $\psi(x)$ contains no terms involving x , $[x, b] = 0$ for all x . Hence $\varphi(x) = \alpha x^{\delta^p}$, as desired.

If R is a field, then $R = Q = R_F$ and also inner derivations of R are trivial. So Problem 1 follows from Theorem 2 as a special instance.

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