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THE SEMISIMPLICITY PROBLEM FOR p-ADIC GROUP ALGEBRAS

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ABSTRACT. For a prime p let $\Omega = \Omega_p$ denote the completion of the algebraic closure of the field of p-adic numbers with p-adic valuation $|\cdot|$. Given a group G consider the ring of formal sums

$$l_1(\Omega, G) = \left\{ \sum_{x \in G} \alpha_x x \colon \alpha_x \in \Omega, |\alpha_x| \to 0 \right\}.$$

Motivated by the study of group rings and the complex Banach algebras $l_1(\mathbf{C}, G)$, we consider the problem of when this ring is semisimple (semiprimitive). Our main result is that for an Abelian group G, $l_1(\Omega, G)$ is semisimple if and only if G does not contain a C_{p^∞} subgroup. We also prove that $l_1(\Omega, G)$ is semisimple if G is a nilpotent p'-group, an ordered group, or a torsion-free solvable group. We use a mixture of algebraic and analytic methods.

I. INTRODUCTION

Throughout p is a fixed prime, and all fields are contained in $\Omega = \Omega_p$, the completion of the algebraic closure of the field of p-adic numbers $\mathbf{Q}_p$, and contain $\mathbf{Q}_p$. If k is such a complete field denote by $|\cdot| \colon k \to \mathbf{R}$ the non-archimedean extension of the p-adic valuation on $\mathbf{Q}_p$. Given a group G consider the ring of formal sums

$$l_1(k, G) = \left\{ \sum_{x \in G} \alpha_x x \colon \alpha_x \in k, |\alpha_x| \to 0 \right\}$$

where $|\alpha_x| \to 0$ means that for every $\varepsilon > 0$, only finitely many of the α_x satisfy $|\alpha_x| \geq \varepsilon$. We are interested in the problem of whether the ring, $l_1(k, G)$, is semisimple (sometimes called semiprimitive) or not. Our main results are that for Abelian groups G, $l_1(k, G)$ is semisimple if G does not contain a C_{p^∞}

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subgroup (§3), and that for a large class of solvable groups including all nilpotent p'-groups, $l_1(\Omega, G)$ is again semisimple (§4).

The algebra $l_1(\mathbf{Q}_p, G)$ was studied in [10]. They consider the question of when $l_1(\mathbf{Q}_p, G)$ is Noetherian, Artinian, or prime and ask when it is semiprime. Our work was also motivated by the well-known result that the Banach algebra over $\mathbf{C}$

$$l_1(G) = \left\{ \sum_{x \in G} \alpha_x x : \alpha_x \in \mathbf{C}, \sum |\alpha_x| < \infty \right\}$$

(here $|\cdot|$ denotes the usual absolute value on $\mathbf{C}$) is always semisimple (cf. [8] or [13]). In contrast it is known that $l_1(\Omega, C_{p^\infty})$ is not semisimple [4].

We conclude the paper with a list of some open problems.

II. General results

Define $\|\cdot\|: l_1(k, G) \to \mathbf{R}$ as follows: if $f = \sum \alpha_x x \in l_1(k, G)$ then $\|f\| = \sup_{x \in G} |\alpha_x|$. Also, write $\operatorname{supp}(f) = \{x \in G : \alpha_x \neq 0\}$. In the following lemma we collect a number of (very easy to prove) properties of $l_1(k, G)$:

2.1. **Lemma.** *Let k be a complete subfield of Ω containing $\mathbf{Q}_p$ and let G be a group. Then*:

(i) *If $f \in l_1(k, G)$ then $\operatorname{supp}(f)$ is countable.*

(ii) $\|f\| = \max_{x \in G} |\alpha_x|$ *if $f = \sum \alpha_x x$, and we have*:
$\|f\| \geq 0$ *for all f, and $\|f\| = 0$ if and only if $f = 0$;*
$\|f + g\| \leq \max\{\|f\|, \|g\|\}$ *and* $\|f + g\| = \|f\|$ *if* $\|g\| < \|f\|$;
$\|fg\| \leq \|f\|\|g\|$.
In particular the topology defined by $\|\cdot\|$ turns $l_1(k, G)$ into an ultrametric topological ring which is complete.

(iii) *The maximal right (or left ideals) of $l_1(k, G)$ are closed.*

(iv) *Let $\varphi: l_1(k, G) \to \Omega$ be a k-algebra homomorphism. Then φ is continuous if and only if $|\varphi(f)| \leq \|f\|$ for all $f \in l_1(k, G)$.*

(v) *Let $\varphi: G \to \Omega$ be a k-algebra homomorphism where $|\varphi(x)| = 1$ for all $x \in G$. Then φ has a unique continuous extension to $l_1(k, G)$.*

Proof. (i) and (ii) are trivial. As for (iii), let M be a maximal right ideal of $l_1(k, G)$. Then its closure $\overline{M}$ is also a right ideal. If $1 \in \overline{M}$ then we can find a sequence $\{f_n\}$ of elements of M with $f_n \to 1$. But if $\|1 - f_n\| < 1$ then $f_n = 1 - (1 - f_n)$ is a unit (with inverse $\sum_{i=0}^{\infty} (1 - f_n)^i \in l_1(k, G)$ since $l_1(k, G)$ is complete). This is impossible, so $1 \notin \overline{M}$ and so $\overline{M} = M$.

(iv) Assume φ is continuous and suppose there exists $x \in G$ with $|\varphi(x)| \neq 1$. Choose $\alpha \in k$ with $0 < |\alpha| < 1$. Replacing x by a suitable power we may assume that $|\varphi(x)| > \alpha^{-1}$. Consider the element $f = \sum_{i=0}^{\infty} \alpha^i x^i \in l_1(k, G)$. If $f_n = \sum_{i=0}^{n} \alpha^i x^i$ then clearly $\|f - f_n\| \to 0$, while $|\varphi(f_n) - \varphi(f_{n-1})| = |\alpha^n \varphi(x)^n| \to \infty$, so the sequence $\{\varphi(f_n)\}$ is not convergent in Ω. This contradicts the continuity of φ. Thus $|\varphi(x)| = 1$ for all $x \in G$. Let $f = \sum_{x\in G} a_x x \in l_1(k, G)$. By continuity $\varphi(f) = \sum a_x \varphi(x)$, thus $|\varphi(f)| \le \|f\|$. The other direction is trivial.

(v) It is easy to check that the unique continuous extension is given by $\varphi(\sum a_x x) = \sum a_x \varphi(x)$. □

If R is a ring we write $J(R)$ for the Jacobson radical of R. The following result enables us to concentrate on countable groups:

2.2. **Lemma.** *If k is a complete field and G is a group, then*

$$J(l_1(k, G)) \subseteq \bigcup_H J(l_1(k, H)),$$

where the union ranges over all countable subgroups H of G.

Proof. Let $f \in J(l_1(k, G))$ and let $H = \langle \mathrm{supp}(f) \rangle$. We claim that $f \in J(l_1(k, H))$. If $g \in l_1(k, H)$ then there exists $f' \in l_1(k, G)$ with $f'(1 - gf) = 1$. Let T be a left transversal of H to G containing 1, and write $f' = \sum t f'_t$, where $f'_t \in l_1(k, H)$. It follows from $\sum_{t\in T} t f'_t(1 - gf) = 1$ that $f'_1(1 - gf) = 1$. This is for all $g \in l_1(k, H)$, so $f \in J(l_1(k, H))$, as required. □

If k is a field write X_k for the class of all groups G for which $l_1(k, G)$ is semisimple. Not much is known about the class-theoretic properties of X_k. As an example we have:

2.3. **Theorem.** *Let k be a complete field, and let $\{N_\lambda : \lambda \in \Lambda\}$ be a directed system of normal subgroups of G such that every $G/N_\lambda \in X_k$. Then $G \in X_k$.*

Proof. Let $0 \ne f \in l_1(k, G)$, and write $f = \sum_{i=1}^r \alpha_i x_i + g$ where $|\alpha_i| = \|f\|$ for all $i = 1, \ldots, r$ and $\|g\| < \|f\|$. By assumption there exists $\lambda \in \Lambda$ with $x_i x_j^{-i} \notin N_\lambda$ for $1 \le i \ne j \le r$. The mapping $\theta: l_1(k, G) \to l_1(k, G/N_\lambda)$ obtained by extending the natural homomorphism $G \to G/N_\lambda$ is easily seen to be a well-defined surjective continuous ring homomorphism. Now $\theta(f) = \sum \alpha_i \theta(x_i) + \theta(g)$, the elements $\theta(x_i)$ are distinct, and $\|\theta(g)\| \le \|g\| < \|f\|$. Thus $\theta(f) \ne 0$, and since $J(l_1(k, G/N_\lambda)) = \{0\}$ this implies that $f \notin J(l_1(k, G))$, as required. □

2.4. **Corollary.** *If G is residually finite then $G \in X_k$ for all k.*

Proof. If $N \lhd_f G$ then $l_1(k, G/N)$ is simply the group ring of G/N over k, which is semisimple by Maschke's Theorem ([11], 2.4.2). □

As another example we have

2.5. **Theorem.** *Let k be complete, with $\bar{k}$ the residue class field of k. If the group ring $\bar{k}G$ is semisimple and has no zero divisors then $G \in X_k$.*

Proof. Suppose $J = J(l_1(k, G)) \neq 0$, so it contains an element g with $\|g\| = 1$. Choose any $f \in l_1(k, G)$ with $\|f\| = 1$. If $\|1 - fg\| < 1$ then $fg = 1 - (1 - fg)$ is invertible, which is impossible since $g \in J$. Let $u \in l_1(k, G)$ be such that $u(1 - fg) = 1$. We claim that $\|u\| = 1$. First, note that

$$1 = \|u(1 - fg)\| \leq \|u\|\|1 - fg\| = \|u\|.$$

Now set

$$R = \{y \in l_1(k, G) \colon \|y\| \leq 1\},$$
$$M = \{y \in l_1(k, G) \colon \|y\| < 1\}.$$

Then M is an ideal of R and $R/M \cong \overline{k}G$ (for the coefficients of the elements of R belong to the valuation ring of k, and the homomorphism onto $\overline{k}$ is easily seen to extend to one of R onto $\overline{k}G$ with kernel M). Let $\pi \colon R \to R/M$ denote the canonical map. If $\|u\| > 1$ choose $\alpha \in k$ with $|\alpha| = \|u\|$. Then $(\alpha^{-1}u)(1 - fg) = \alpha^{-1}$, and so $(\alpha^{-1}u)\pi(1 - fg)\pi = \alpha^{-1}\pi = 0$. But this is impossible since $\|\alpha^{-1}u\| = \|1 - fg\| = 1$ and $\overline{k}G$ has no zero divisors. Thus $\|u\| = 1$. But then $(u\pi)(1 - f\pi g\pi) = 1$, and since this is true for all $f \in l_1(k, G)$ it follows that $g\pi \in J(\overline{k}G) = \{0\}$, contradicting $\|g\| = 1$. □

2.6. **Corollary.** *Let k be a complete field.*

(i) *If G is a u.p. group (e.g. an ordered group) then $G \in X_k$.*
(ii) *If $\overline{k}$ is uncountable and $\overline{k}G$ has no zero divisors then $G \in X_k$.*

Proof. These are consequences of 2.5 and well-known facts about the group-ring $\overline{k}G$ (cf. [11], 13.1.2, 13.1.9, 7.1.6) in view of the fact that u.p. groups are t.u.p. groups [14]. □

2.7. **Corollary.** *Let k be complete with residue class field $\overline{k}$, and let G and H be nontrivial groups. If the group ring $\overline{k}(G \times H)$ has no zero divisors then $G \times H \in X_k$.*

Proof. The assumptions imply that $\overline{k}(G \times H)$ is semisimple [9]. The result follows from 2.5. □

2.8. **Corollary.** *Let k be complete with residue class field $\overline{k}$. Suppose G is a torsion-free solvable group. Then $G \in X_k$.*

Proof. In [7] it is shown that for such groups G, $\overline{k}(G)$ has no zero divisors. It is well known that $\overline{k}(G)$ is semisimple (cf. [11] 7.4.6). □

III. Abelian groups

For $f \in l_1(k, G)$ let $v(f) = \lim_{n\to\infty} \|f^n\|^{1/n}$ (it is easy to see that $v(f)$ is well defined, [12] 6.22). Say f is *topologically nilpotent* if $v(f) = 0$. Our aim is to show that for Abelian groups G, the Jacobson radical of $l_1(k, G)$ is precisely the set of topological nilpotents in $l_1(k, G)$, as is the case for the complex algebra $l_1(G)$ [8]. We will write $\Phi_k(G)$ for the set of all continuous k-algebra homomorphisms $\varphi: l_1(k,G) \to \Omega$ and write $\Phi'_k(G)$ for the set of $\varphi \in \Phi_k(G)$ such that $\varphi(l_1(k,G))$ is a field. We have:

3.1. **Lemma.** *Let G be an Abelian group, with H a subgroup of G. Then any element of $\Phi_k(H)$ (respectively, $\Phi'_k(H)$) can be extended to an element of $\Phi_k(G)$ (respectively, $\Phi'_k(G)$).*

Proof. Apply Zorn's Lemma to the set of all triplets (G_1, φ, E) where $G_1 \supseteq H$ is a subgroup of G, $\varphi \in \Phi_k(G_1)$, (respectively, $\Phi'_k(G_1)$), and $E = \varphi(l_1(k,G_1))$. It is routine to show that G belongs to a maximal element (cf. [11] 1.2.7). $\square$

We need to introduce an auxiliary seminorm on $l_1(k,G)$: if $f \in l_1(k,G)$ let

$$\|f\|_{sp} = \sup\{|\varphi(f)|: \varphi \in \Phi'_k(G)\}.$$

We have the following result:

3.2. **Lemma.** *Let k be a complete field, and let G be an Abelian group.*

(a) $v(f+g) \le \max\{v(f),\ v(g)\}$ *for all* $f,\ g \in l_1(k,G)$.
(b) $v(f) \le \|f\|$ *for all* $f \in l_1(k,G)$.
(c) v *is continuous.*
(d) $\|f\|_{sp} \le v(f) \le p\|f\|_{sp}$ *for all* $f \in l_1(k,G)$.

Proof. For (a) see [12], 6.22. Part (b) is obvious. Part (c) follows easily from (a) and (b). Consider (d). The inequality $\|f\|_{sp} \le v(f)$ follows from $\|f^n\|_{sp} \le \|f^n\|$. To prove the second inequality, first consider the case when G is finitely generated, and let $\lambda \in k$ satisfy $\|f\|_{sp} < |\lambda|$. Then $|\varphi(\lambda^{-1}f)| = |\lambda^{-1}||\varphi(f)| \le |\lambda|^{-1}\|f\|_{sp} < 1$ for all $\varphi \in \Phi'_k$, and so $(\lambda^{-1}f)^n \to 0$ as $n \to \infty$ [15]. Thus for all sufficiently large n we have $\|(\lambda^{-1}f)^n\| < 1$, whence $\|f^n\|^{1/n} < |\lambda|$. Thus $v(f) \le |\lambda|$ whenever $\lambda \in k$ satisfies $\|f\|_{sp} < |\lambda|$. Since $|k^*| \supseteq \{p^n: n \in \mathbf{Z}\}$, and in particular has 0 as an accumulation point, it follows that $v(f) = 0$ if $\|f\|_{sp} = 0$. It also follows that if $\|f\|_{sp} \neq 0$ then there exists $\lambda \in k$, with $\|f\|_{sp} < |\lambda| \le p\|f\|_{sp}$, and hence $v(f) \le p\|f\|_{sp}$.

In the general case write $f = \lim_{n\to\infty} f_n$ where each $f_n \in l_1(k,G)$ has finite support in G. Given $\varepsilon > 0$ choose N large enough so that $\|f - f_N\| < \varepsilon$, and $|v(f_N) - v(f)| < \varepsilon$. Choose $\psi \in \Phi'_k(G)$ such that $|\psi(f_N)| > p^{-1}v(f_N) - \varepsilon$. (The existence of ψ follows from the first part of the argument applied to $\langle \mathrm{supp}(f_N)\rangle$ and 3.1.) Thus

$$\begin{aligned}\|f\|_{sp} &\ge |\psi(f)| \ge |\psi(f_N)| - |\psi(f - f_N)| > p^{-1}v(f_N) - \varepsilon - \|(f - f_N)\| \\ &\ge p^{-1}v(f) - p^{-1}|v(f_N) - v(f)| - 2\varepsilon > p^{-1}v(f) - (2 + p^{-1})\varepsilon.\end{aligned}$$

Since ε is arbitrary, the result follows. $\square$

The proof shows that when $|k^*|$ is dense in the positive real numbers, then we obtain $v(f) \le \|f\|_{sp}$, and so in fact $v(f) = \|f\|_{sp}$.

We can now prove the following:

3.3. **Theorem.** *Let k be a complete field, G an Abelian group, and $f \in l_1(k, G)$. Then the following are equivalent:*

(a) $f \in J(l_1(k, G))$.

(b) $\varphi(f) = 0$ *for all* $\varphi \in \Phi'_k(G)$.

(c) f *is topologically nilpotent.*

(d) $\varphi(f) = 0$ *for all* $\varphi \in \Phi_k(G)$.

Proof.

$(a) \Rightarrow (b)$ is trivial, since $\ker \varphi$ is a maximal ideal of $l_1(k, G)$.

$(b) \Rightarrow (c)$ follows from 3.2 since (b) implies that $\|f\|_{sp} = 0$.

$(c) \Rightarrow (a)$: let $g \in l_1(k, G)$. Since $\|(gf)^n\|^{1/n} \le \|g\|\|f^n\|^{1/n} \to 0$ we have $\|(gf)^n\| \to 0$, so $\sum_{n=0}^{\infty}(gf)^n \in l_1(k, G)$. In other words $1 - gf$ is invertible, whence $f \in J(l_1(k, G))$.

$(c) \Rightarrow (d)$: a continuous homomorphism must map topological nilpotents to topological nilpotents, and the only topological nilpotent in a field is the zero element.

$(d) \Rightarrow (b)$: Obvious. □

The following result is useful for dealing with extensions:

3.4. **Lemma.** *Let G be Abelian, H a subgroup of G, and let k be a complete field. Assume that $H \in X_k$ and $G/H \in X_E$ for all complete extension fields $E \supseteq k$. Then $G \in X_k$.*

Proof. Let $f = \sum_{x \in G} \alpha_x x \in J(l_1(k, G))$. Let $\varphi \in \Phi'_k(H)$, and let $\theta \in \Phi'_k(G)$ be any extension of φ (such exist by 3.1). Put $E = \theta(l_1(k, G))$, a subfield of Ω. Let $x \mapsto \overline{x}$ denote the natural map $G \to G/H$. If $\mu \in \Phi_{\overline{E}}(G/H)$, where $\overline{E}$ is the completion of E, consider the map $\psi: l_1(k, G) \to \Omega$ defined by $\psi(x) = \theta(x)\mu(\overline{x})$ for all $x \in G$, and extended by linearity and continuity to the whole space. Since $\psi \in \Phi_k(G)$, $\psi(f) = 0$ by 3.3. Put $\beta_x = \alpha_x \theta(x) \in E$, and $\gamma_{\overline{x}} = \sum_{\overline{y} = \overline{x}} \beta_y \in E$. In view of the definition of ψ we have $0 = \psi(f) = \sum_{\overline{x} \in G/H} \gamma_{\overline{x}} \mu(\overline{x}) = \mu(\sum_{\overline{x}} \gamma_{\overline{x}} \overline{x})$. This is for all $\mu \in \Phi_{\overline{E}}(G/H)$, and $l_1(\overline{E}, G/H)$ is semisimple by assumption, so by 3.3 again, $\gamma_{\overline{x}} = 0$ for all $\overline{x} \in G/H$. But $\overline{y} = \overline{x}$ is equivalent to $y \in Hx$ so we have $\sum_{h \in H} \alpha_{hx} \theta(hx) = 0$, where $x \in G$ is fixed. Cancelling a factor of $\theta(x)$, and remembering that θ is an extension of φ, the above becomes $\varphi(\sum_{h \in H} \alpha_{hx} h) = 0$. But this is for all $\varphi \in \Phi'_k(H)$, so finally every $\alpha_{hx} = 0$, as required. □

We need one more preliminary result.

3.5. **Lemma.** *Let G be an Abelian p'-group, and let k be a complete subfield of Ω. Given distinct elements $x_1, \dots, x_n$ of G and elements $c_1, \dots, c_n$*

of k, there exists a continuous k-homomorphism $\varphi: l_1(k, G) \to \Omega$ such that $|\varphi(\sum_{i=1}^n c_i x_i)| = \| \sum_i c_i x_i \|$.

Proof. Let $f = \sum c_i x_i$. We may assume that $\|f\| = 1$. If we can find a continuous φ such that $|\varphi(f)| \geq 1$ then the trivial fact $|\varphi(f)| \leq \|f\|$ suffices to show that $|\varphi(f)| = 1 = \|f\|$.

The proof of $|\varphi(f)| \geq 1$ proceeds by induction on n, the case $n = 1$ being trivial (let $\varphi(x) = 1$ for all $x \in G$). Assume the result is true for $n-1$ but false for n. Thus there exist distinct elements $x_1, \dots, x_n \in G$, and $c_1, \dots, c_n \in k$ with $\max_i |c_i| = 1$, such that $|\varphi(f)| < 1$ for all $\varphi \in \Phi_k(G)$ (where $f = \sum c_i x_i$). We may suppose $|c_n| = 1$. It is easy to see that if $\lambda \neq 1$ is a p'-power root of unity in Ω then $|1 - \lambda| = 1$. Since $x_1^{-1} x_n \neq 1$ we can find a p'-power root of unity $\lambda \in \Omega$, $\lambda \neq 1$, of the same order as $x_1^{-1} x_n$. (If the order of $x_1^{-1} x_n$ is infinite let $\lambda \neq 1$ be any p'-power root of unity.) Define a continuous k-algebra homomorphism $\varphi: l_1(k, \langle x_1^{-1} x_n \rangle) \to \Omega$ by $\varphi(x_1^{-1} x_n) = \lambda$. Extend this to an element (still denoted by φ) of $\Phi_k(G)$. Then

$$|\varphi(x_1) - \varphi(x_n)| = |\varphi(x_1)||1 - \varphi(x_1^{-1} x_n)| = 1,$$

and so $\max_i |c_i(\varphi(x_1) - \varphi(x_i))| = 1$. By the inductive hypothesis there exists a continuous homomorphism $\mu: l_1(k, G) \to \Omega$ such that

$$(*) \qquad \left| \sum_{i=1}^n c_i(\varphi(x_1) - \varphi(x_i))\mu(x_i) \right| \geq 1.$$

The product homomorphism $\varphi\mu$ (defined by $\varphi\mu(\sum a_x x) = \sum a_x \varphi(x)\mu(x)$) is also continuous, and so

$$\left| \sum_{i=1}^n c_i \varphi(x_i)\mu(x_i) \right| < 1.$$

Since $|\varphi(x_1)| = 1$ we also have

$$\left| \sum_{i=1}^n c_i \varphi(x_1)\mu(x_i) \right| = \left| \sum_{i=1}^n c_i \mu(x_i) \right| < 1,$$

and thus

$$\left| \sum_{i=1}^n c_i(\varphi(x_1) - \varphi(x_i))\mu(x_i) \right| < 1.$$

This contradicts $(*)$, and proves the result. □

Our main result on Abelian groups is

3.6. **Theorem.** *Let k be a complete subfield of Ω_p, and let G be an Abelian group with no C_{p^∞} subgroups. Then $G \in X_k$.*

Proof. We proceed via a series of steps.

Step 1. We may assume that G is a p-group: If G_p denotes the maximal p-subgroup of G and E is a complete extension field of k, then we claim that

$G/G_p \in X_E$. For if $f \neq 0$ is an element of $l_1(E, G/G_p)$ then by 3.5 we can find a continuous homomorphism $\varphi \in \Phi_E(G/G_p)$ such that $|\varphi(f)| = \|f\| \neq 0$, so $J(l_1(E, G/G_p)) = \{0\}$ by 3.3. Thus if $G_p \in X_k$, then $G \in X_k$ by 3.4. We are now in the case where G is a countable (2.2) Abelian p-group with no $C_{p\infty}$ subgroup.

Step 2. If G has finite exponent p^m then $G \in X_k$: If G has exponent p then it is a countable vector space over $GF(p)$, and is therefore residually finite. Thus $G \in X_k$ by 2.4. In general by induction on m we have $G/G^{p^{m-1}}$ and $G^{p^{m-1}} \in X_k$ (for all fields k), and so $G \in X_k$ by 3.4.

Step 3. If G has no element of infinite height then $G \in X_k$: The assumption is that $\bigcap_{m=1}^{\infty} G^{p^m} = \langle 1 \rangle$, so $\{G^{p^m} : m = 1, 2, \ldots\}$ is a directed system in G. Each $G/G^{p^m} \in X_k$ by Step 2, and so $G \in X_k$ by 2.3.

We can now deal with countable reduced p-groups G. Consider the Ulm sequence of G ([6], §76): put $G^{(1)} = \bigcap_{n=1}^{\infty} G^{p^n}$; if σ is not a limit ordinal put $G^{\sigma+1} = (G^{(\sigma)})^{(1)}$, and if λ is a limit ordinal put $G^{(\lambda)} = \bigcap_{\sigma<\lambda} G^{(\sigma)}$. Since G is reduced we have $G^{(\tau)} = \langle 1 \rangle$ for some ordinal τ. We prove, by transfinite induction on σ, that $G/G^{(\sigma)} \in X_k$ for all σ. For $\sigma = 1$ the group $G/G^{(1)}$ has no elements of infinite height and so $G/G^{(1)} \in X_k$ by Step 3. If σ is not a limit ordinal then $G^{(\sigma-1)}/G^{(\sigma)} \in X_k$ by Step 3, and $G/G^{(\sigma-1)} \in X_E$ (for all complete E) by induction, so $G/G^{(\sigma)} \in X_k$ by 3.4. If σ is limit ordinal then $\{G^{(\rho)}/G^{(\sigma)} : \rho < \sigma\}$ is a directed system in $G/G^{(\sigma)}$, and since each $G/G^{(\rho)} \in X_k$ by induction, we have $G/G^{(\sigma)} \in X_k$ by 2.3. This establishes the inductive step. In particular $G = G/G^{(\tau)} \in X_k$, as required. □

In general $l_1(k, C_{p\infty})$ is semisimple if and only if k does not contain infinitely many pth-power roots of unity ([4]; see also [1], [5]). In fact, if k^* contains a $C_{p\infty}$ subgroup then $l_1(k, C_{p\infty})$ even contains nonzero nilpotent elements. Thus no improvement to 3.6 without imposing additional restrictions on the field k is possible and the following corollary is obvious:

3.7. **Corollary.** *If G is an Abelian group, then $l_1(\Omega, G)$ is semisimple if and only if G does not contain a $C_{p\infty}$ subgroup.*

IV. Solvable groups

To obtain results for solvable groups we need another extension theorem.

4.1. **Lemma.** *Let G be an Abelian p'-group and suppose $\{x_i\}_{i=1}^{\infty} \subset G$ with $x_i \neq x_j$ for $i \neq j$. Then there exists $\varphi_i \in \Phi_\Omega(G)$ such that if $A_n = (\varphi_i(x_j))_{i,j=1}^{n}$, then*

(1) $$|\det A_n| = 1 \quad \text{for all } n, \text{ and}$$

(2) $$\text{if } A_n^{-1} = (b_{ij}^{(n)})_{i,j=1}, \text{ then } |b_{ij}^{(n)}| \leq 1 \quad \text{for } i, j = 1, \ldots, n.$$

Remark. This lemma can be compared to results on k-complete group algebras (c.f. [11], 4.3.3).

Proof. The proof proceeds by induction on n, the case $n = 1$ being trivial (take $\varphi_1 \equiv 1$). Assume the result holds for $n-1$, so $\varphi_1, \dots, \varphi_{n-1} \in \Phi_\Omega(G)$ have been found satisfying properties (1) and (2). Let $\mu \in \Phi_\Omega(G)$ and consider the $n \times n$ matrix $M(\mu) = (a_{ij})$ where

$$a_{ij} = \begin{cases} \varphi_i(x_j) & \text{if } i = 1, \dots, n-1, \quad j = 1, \dots, n \\ \mu(x_j) & \text{if } i = n, \quad j = 1, \dots, n. \end{cases}$$

Expanding the determinant along the nth row we obtain $\det M(\mu) = \sum_{i=1}^n \mu(x_i)c_i$, where $|c_i| \le 1$ for $i = 1, \dots, n-1$ and $c_n = (-1)^{n-1} \det A_{n-1}$. By the induction assumption $|c_n| = 1$, thus by 3.5 there exists some $\varphi_n \in \Phi_\Omega(G)$ such that $|\sum_{i=1}^n c_i \varphi_n(x_i)| = 1$. Thus (1) is satisfied. Moreover the cofactor formula for inverses makes it clear that $|b_{ij}^{(n)}| \le 1$ for $i, j = 1, \dots, n$ establishing (2).

4.2. **Theorem.** *Let H be a normal subgroup of a group G and suppose G/H is an Abelian p'-group. If $f \in J(l_1(\Omega, G))$ then $f = \sum_{i=1}^\infty t_i x_i$ where $t_i \in J(l_1(\Omega, H))$, $x_i \in G$ and $\|t_i\| \to 0$.*

Proof. Let $f \in J(l_1(\Omega, G))$ and suppose $f = \sum_{i=1}^\infty f_i$ with $f_i \in l_1(\Omega, G)$ and $\operatorname{supp} f_i \subseteq Hx_i$ where $\{x_i\}_{i=1}^\infty$ are distinct coset representatives of G/H. Let $\pi\colon G \to G/H$ be the canonical quotient map. Apply 4.1 with the Abelian p'-group being G/H and $\{x_i\pi\}$ the distinct elements of G/H, to obtain $\varphi_i \in \Phi_\Omega(G/H)$ with the corresponding properties (1) and (2). Extend each φ_i to elements of $\Phi_\Omega(G)$ by defining

$$\varphi_i\left[\sum_{g\in G} a_g g\right] = \sum_{g \in G} a_g \varphi_i(g\pi).$$

Define

$$\varphi_i^{\#}\left[\sum a_g g\right] = \sum a_g \varphi_i(g) g.$$

Then φ_i^* is an Ω-automorphism of $l_1(\Omega, G)$. Notice that as $\operatorname{supp} f_j \subset Hx_j$

$$(*) \qquad \varphi_i^*(f) = \sum_{j=1}^\infty \varphi_i(x_j) f_j.$$

Let $\varepsilon > 0$ and choose N such that $\|f_n\| < \varepsilon$ for $n > N$. Let

$$B_N = (\varphi_i(x_{N+j})) \qquad \text{where } i = 1, \dots, N \text{ and } j = 1, 2, \dots.$$

From $(*)$ we have the system of equations

$$(\varphi_i^{\#}(f))_{i=1,\dots,N} = A_N(f_i)_{i=1,\dots,N} + B_N(f_i)_{i=N+1,N+2,\dots}.$$

Hence

$$(f_i)_{i=1,\dots,N} + A_N^{-1} B_N (f_i)_{i=N+1,N+2,\dots} = A_N^{-1}(\varphi_i^{\#}(f))_{i=1,\dots,N}.$$

As $J(l_1(\Omega, G))$ is invariant under Ω-automorphisms, $\varphi_i^{\#}(f) \in J(l_1(\Omega, G))$. Let

$$(j_i^{(N)})_{i=1,\dots,N} = A_N^{-1}(\varphi_i^{\#}(f))_{i=1,\dots,N}.$$

Then $j_i^{(N)}$ is a linear combination of $\varphi_1^{\#}(f), \dots, \varphi_N^{\#}(f)$ and thus belongs to the Jacobson radical for each integer N and $i = 1, \dots, N$. Finally, let

$$(\varepsilon_i^{(N)})_{i=1,\dots,N} = A_N^{-1} B_N (f_j)_{j=N+1,N+2,\dots}.$$

Then $f_i = j_i^{(N)} + \varepsilon_i^{(N)}$. As all the entries of A_N^{-1} and B_N have valuation at most one, (by the lemma) and $\|f_n\| < \varepsilon$ for all $n > N$, it follows that $\|\varepsilon_i^{(N)}\| < \varepsilon$ for all $i = 1, \dots, N$. As $\varepsilon > 0$ was arbitrary we have that, for each i, $\lim_{n\to\infty} \|j_i^{(N)} - f_i\| = 0$ and as $J(l_1(\Omega, G))$ is closed (2.1(iii)) $f_i \in J(l_1(\Omega, G))$. Thus

$$f_i x_i^{-1} \in J(l_1(\Omega, G)) \cap l_1(\Omega, H) \subseteq J(l_1(\Omega, H))$$

(c.f. the proof of 2.2) and $f = \sum (f_i x_i^{-1}) x_i$. $\square$

The next result is obvious.

4.3. **Corollary.** *Suppose G_1 is an Abelian p'-group and $G_2 \in X_\Omega$. Then $G_1 \times G_2 \in X_\Omega$.* $\square$

Let $S_{p'}$ denote the class of solvable groups which have a subnormal series with Abelian p'-factor groups. This class contains all solvable torsion groups which have no elements of order p and all nilpotent p'-groups [2].

4.4. **Theorem.** *Suppose H is a normal subgroup of G with $H \in X_\Omega$ and $G/H \in S_{p'}$. Then $G \in X_\Omega$.*

Proof. Let

$$\{1\} = K_1 \le K_2 \le \cdots \le K_N = G/H$$

be a subnormal series for G/H with K_{i+1}/K_i a p'-Abelian group for each $i = 1, \dots, N$. Let $\pi^{-1}(K_i) = G_i$. We have

$$H = G_1 \le G_2 \le \cdots \le G_n = G$$

with $G_{i+1}/G_i \simeq K_{i+1}/K_i$ p'-Abelian groups, and as $H \in X_\Omega$ an induction argument together with Theorem 4.2 now completes the proof. $\square$

4.5. **Corollary.** *Let $G \in S_{p'}$. Then $G \in X_\Omega$.* $\square$

4.6. **Corollary.** *Let G have a directed system $\{N_i : i \in I\}$ such that each $G/N_i \in S_{p'}$. Then $G \in X_\Omega$.*

Proof. Combine Theorem 2.3 and Corollary 4.5. $\square$

4.7. **Theorem.** *Let G have a directed system $\{N_i : i \in I\}$ such that each factor group G/N_i is polycyclic. Then $G \in X_\Omega$.*

Proof. As usual we may assume G itself is polycyclic, so G has a subnormal series

$$G = G_n \ge G_{n-1} \ge \cdots \ge G_0 = \{1\}$$

with each G_i/G_{i-1} a finitely generated Abelian group. The proof proceeds by induction on n. If $n = 1$ the result is clear, as finitely generated Abelian groups do not contain C_{p^∞}. So assume $G_{i-1} \in X_\Omega$.

Choose a subgroup L of G with $G_i \geq L \geq G_{i-1}$, G_i/L a torsion-free Abelian group and L/G_{i-1} a finite Abelian group. Let $\pi: L \to L/G_{i-1}$ be the usual quotient map and assume $L/G_{i-1} = \{x_i\pi\}_{i=1}^N$, $x_i \in L$, with $x_i\pi$ distinct. By [11] 4.3.3 there exist homomorphisms $\varphi_1, \ldots, \varphi_N \in \Phi_\Omega(L/G_{i-1})$ such that the $N\times N$ matrix $(\varphi_i(x_j\pi))_{i,j=1}^N$ is nonsingular. Using this result in place of 4.1, arguments similar to those of the proof of 4.2, (but much easier as the matrix B_N is unnecessary) show that if $f \in J(l_1(\Omega, L))$ then $f = \sum_{i=1}^N t_i x_i$ with $t_i \in J(l_1(\Omega, G_{i-1})) = (0)$. Thus $L \in X_\Omega$. As G_i/L is an Abelian p'-group, $G_i \in X_\Omega$ (4.2). This completes the induction step and hence the proof. □

Open problems

1. Is $l_1(\mathbf{Q}_p, \oplus_1^N C_{p^\infty})$ semisimple? If so then it can be proved that $l_1(\mathbf{Q}_p, G)$ is semisimple for any Abelian group G.

2. If G is a nilpotent group which does not contain a C_{p^∞} subgroup, is $l_1(\Omega, G)$ semisimple?

3. Is the class X_k closed under direct products or (normal) subgroups?

4. If $G \in X_\Omega$ and k is any complete extension field of $\mathbf{Q}_p$ contained in Ω, is $G \in X_k$?

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