PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 108, Number 3, March 1990

NORMS OF HANKEL OPERATORS ON A BIDISC

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(Communicated by Paul S. Muhly)

ABSTRACT. In the Hardy space on the bidisc T^2, if ϕ is a bounded function in the Lebesgue space and if its Fourier series vanishes on half of $\mathbf{Z}^2$, then the norm of the Hankel operator H_ϕ is equal to the quotient norm of ϕ by the Hardy space $H^\infty(T^2)$.

Let σ denote the Haar measure of the torus T^2, the distinguished boundary of the unit bidisc U^2 in the space of 2 complex variables (z, w). Put $\mathbf{Z}_+^2 = \{(m, n) \in \mathbf{Z}^2; m \geq 0 \text{ and } n \geq 0\}$. For $1 \leq p \leq \infty$, $L^p = L^p(T^2, \sigma)$ denotes the Lebesgue space and $H^p = H^p(T^2, \sigma) = \{f \in L^p; \hat{f}(m, n) = 0 \text{ if } (m, n) \notin \mathbf{Z}_+^2\}$. That is, H^p denotes the usual Hardy spaces on the bidisc. Let $K_0^p = \{f \in L^p; \hat{f}(m, n) = 0 \text{ if } (-n, -m) \in \mathbf{Z}_+^2\}$.

For $\phi \in L^\infty$, the Hankel operator determined by ϕ is

$$H_\phi = (I - P)M_\phi | H^2$$

and the Toeplitz operator determined by ϕ is

$$T_\phi = PM_\phi | H^2,$$

where $P: L^2 \to H^2$ is the orthogonal projection and M_ϕ is the multiplication operator on L^2 associated with ϕ.

In the classical Hardy space $H^2(T)$ on the unit disc, the norms of Hankel operators H_ϕ are equal to the quotient norms of the symbols ϕ by $H^\infty(T)$ ([7], [8, pp. 4–6]). This is called Nehari's theorem. The author [6] gave a generalization of Nehari's theorem to the Hardy space $H^2(T^2)$ on the bidisc. The theorem is analogous to Nehari's theorem but does not give the norm of H_ϕ using the symbol ϕ. For the Hardy space on the unit ball of $\mathbf{C}^n$, it is known that the norm of a Hankel operator H_ϕ is equivalent to the BMO norm

Received by the editors March 1, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 47A30, 47B35; Secondary 46J15, 42B30.
Key words and phrases. bidisc, Hardy space, Hankel operator, norm.
This research was partially supported by Grant-in-Aid for Scientific Research, Ministry of Education.

0002-9939/90 \$1.00 + \$.25 per page

of ϕ (cf. [2], [9]). However we do not know whether the same theorem is valid or not for that on the polydisc. In this paper, for the bidisc we show that if $\phi \in L^\infty$ is a function whose Fourier series vanish on the half of $\mathbf{Z}^2$, then $\|H_\phi\| = \|\phi + H^\infty\|$.

In the classical Hardy space, the essential norms of Hankel operators are the quotient norms of the symbols by $H^\infty + C(T)$, where $C(T)$ denotes the set of all continuous functions on the unit circle (cf. [8, p. 6]). Hence there exist many nonzero compact Hankel operators. However Curto, Muhly and Nakazi [3] showed that there does not exist any nonzero compact Hankel operator in the bidisc Hardy space. In this paper, roughly speaking, we show that if $\phi \in L^\infty$ is a function whose Fourier series vanish on the half of $\mathbf{Z}^2$, then $\|H_\phi\|_e = \|H_\phi\|$ where $\|H_\phi\|_e$ denotes the essential norm. Moreover we use results about Hankel operators to study the invertibility of Toeplitz operators.

An order relation can be introduced in $\mathbf{Z}^2$. Let L_r be a line with rational slope r in the plane. S_r denotes all lattice points on one side of L_r, together with those on the right side ray of L_r from the origin. When L is a real axis, that is, $L = L_0$, then $S_0 = \{(m,0): m > 0\} \cup \{(m,n): m > 0\}$. When L is an imaginary axis, that is, $L = L_{-\infty}$, then $S_{-\infty} = \{(m,n): m > 0\} \cup \{(0,n): n > 0\}$. This order is non-archimedean, and $\mathbf{Z}^2$ has a smallest positive element (m_0, n_0) in S_r. We assume that S_r contains $\mathbf{Z}^2_+$, that is, $-\infty \le r \le 0$. When $-\infty < r < 0$, $|m_0|$ and $|n_0|$ have no common factor except 1 and $r = n_0/m_0$, and $(m_1, n_1) = (0,1)$. When $r = 0$, $(m_0, n_0) = (1,0)$ and let $(m_1, n_1) = (0,1)$. When $r = -\infty$, $(m_0, n_0) = (0,1)$ and let $(m_1, n_1) = (1,0)$. For each half plane S_r, put

$$Z = Z_r = z^{m_0} w^{n_0}$$

and

$$W = W_r = z^{m_1} w^{n_1}.$$

Hence $Z_0 = W_{-\infty} = z$ and $W_0 = Z_{-\infty} = w$, and if $-\infty < r < 0$ then $W_r = w$.

For each r with $-\infty \le r \le 0$, put $\mathbf{H}^p_r =$ the norm closure of $\cup_{j=-\infty}^{\infty} Z_r^j H^p$ in L^p if $1 \le p < \infty$ and $\mathbf{H}^\infty_r =$ the weak^{-*} closure of $\cup_{j=-\infty}^{\infty} Z_r^j H^\infty$ in L^∞. $\mathscr{L}^p_r$ and $\mathscr{H}^p_r$ denote the norm closure of the set of trigonometric polynomials and analytic polynomials, respectively, of Z_r in L^p if $1 \le p < \infty$. $\mathscr{L}^\infty_r$ and $\mathscr{H}^\infty_r$ denote the weak^{-*} closure. Then

$$\mathbf{H}^p_r = \mathscr{L}^p_r + \mathscr{L}^p_r W + \cdots + \mathscr{L}^p_r W^{n-1} + W^n \mathbf{H}^p_r.$$

Let E be a conditional expectation from $\mathbf{H}^\infty_r$ onto $\mathscr{L}^\infty_r$. Then E is multiplicative on $\mathbf{H}^\infty_r$ and $\mathbf{H}^\infty_r + \overline{W\mathbf{H}^\infty_r}$ is weak^{-*} dense in L^∞. Hence $\mathbf{H}^\infty_r$ is an extended weak^{-*} Dirichlet algebra with respect to E (see [5]). Thus Theorems 4 and $4'$ in [5] imply

Lemma 1. *Let* $h \in \mathbf{H}_r^1$ *and let* $\varepsilon > 0$ *be given. Then there is an* $f \in \mathbf{H}_r^2$, $\|f\|_2 \le \|h\|_1$, *and a* $g \in \mathbf{H}_r^2$, $\|g\|_2 \le \|h\|_1 + \varepsilon$, *such that* $h = fg$.

The proof is almost the same as the proof of Lemma 2.1 in [4].

Theorem 1. *If* ϕ *is a function in* L^∞, *then*

$$\|\phi + H^\infty\| \ge \|H_\phi\| \ge \sup\{\|\phi + \mathbf{H}_r^\infty\|\,;\, -\infty \le r \le 0\}.$$

Proof. It is sufficient to show that $\|H_\phi\| \ge \|\phi + \mathbf{H}_r^\infty\|$ for any r with $-\infty \le r \le 0$. Fix r with $-\infty \le r \le 0$. Since $Z^j\mathbf{H}_r^2 = \mathbf{H}_r^2$ for any integer j, $H^2 \times W\mathbf{H}_r^2 = Z^jH^2 \times W\mathbf{H}_r^2$ and hence

$$H^2 \times K_0^2 \supset \{\cup_{j=-\infty}^{\infty} Z^jH^2\} \times W\mathbf{H}_r^2.$$

Hence,

$$\begin{aligned}\|H_\phi\| &= \sup\left\{\left|\int \phi f g\, d\sigma\right| : f \in H^2, g \in K_0^2,\right.\\ &\qquad\qquad \left. \|f\|_2 \le 1 \text{ and } \|g\|_2 \le 1\right\}\\ &\ge \sup\left\{\left|\int \phi f g\, d\sigma\right| : f \in \cup_{j=-\infty}^{\infty} Z^jH^2, g \in W\mathbf{H}_r^2,\right.\\ &\qquad\qquad \left. \|f\|_2 \le 1 \text{ and } \|g\|_2 \le 1\right\}\\ &= \sup\left\{\left|\int \phi f g\, d\sigma\right| : f \in \mathbf{H}_r^2, g \in W\mathbf{H}_r^2,\right.\\ &\qquad\qquad \left. \|f\|_2 \le 1 \text{ and } \|g\|_2 \le 1\right\}\\ &= \|\phi + \mathbf{H}_r^\infty\|.\end{aligned}$$

We used Lemma 1 in the last equality (cf. [4, Corollary 2.1.1.]). □

Lemma 2. *If* $\phi \in \overline{W\mathbf{H}_r^\infty}$ *then* $\|\phi + \mathbf{H}_r^\infty\| = \|\phi + H^\infty\|$.

Proof. By duality, it is sufficient to show that

$$\begin{aligned}&\sup\left\{\left|\int \phi f\, d\sigma\right| : f \in K_0^1, \|f\|_1 \le 1\right\}\\ &\qquad\qquad = \sup\left\{\left|\int \phi F\, d\sigma\right| : F \in W\mathbf{H}_r^1, \|F\|_1 \le 1\right\}.\end{aligned}$$

Let Q be an orthogonal projection from L^2 onto $W\mathbf{H}_r^2$. If $\phi \in \overline{W\mathbf{H}_r^\infty}$, then for any $f \in K_0^2$,

$$\int \phi f\, d\sigma = \int \phi Q(f)\, d\sigma.$$

Since $QK_0^2 = W\mathbf{H}_r^2$, the two supremums above are equal because K_0^2 is dense in K_0^1 and $\mathbf{H}_r^2$ is dense in $\mathbf{H}_r^1$. □

Theorem 2. *For any r with $-\infty \le r \le 0$, if ϕ is in $\overline{W\mathbf{H}_r^\infty} + H^\infty$, then*

$$\|H_\phi\| = \|\phi + H^\infty\|.$$

The proof is clear by Theorem 1 and Lemma 2.

For any r with $-\infty \le r \le 0$, we want to know about $\|H_\phi\|$ for ϕ in $\mathscr{L}_r^\infty$. The following proposition answers the request only when $r = 0$ or $-\infty$.

Proposition 3. *Put $r = 0$ or $-\infty$. If ϕ is in $\mathscr{L}_r^\infty + H^\infty$ then*

$$\|H_\phi\| = \|\phi + H^\infty\|.$$

Proof. For any r, $\mathscr{H}_r^p$ is isometrically isomorphic to the classical Hardy space $H^p(T)$. Hence by Nehari's theorem (cf. [8, p. 11]),

$$\begin{aligned}\|H_\phi\| \ge \sup &\left\{ \left| \int \phi f g \, d\sigma \right| ; f \in \mathscr{H}_r^2, g \in \mathscr{H}_r^2, \right.\\ &\left. \qquad\qquad\qquad \|f\|_2 \le 1 \text{ and } \|g\|_2 \le 1 \right\}\\ = \|\phi &+ \mathscr{H}_r^\infty\|.\end{aligned}$$

If $r = 0$ or $-\infty$, $H^\infty \supset \mathscr{H}_r^\infty$ and hence the proposition follows. □

By Theorem 1, we can ask whether $\|H_\phi\| = \sup_r \|\phi + \mathbf{H}_r^\infty\|$ is true or not. By Theorem 2, if $\phi \in \overline{W\mathbf{H}_r^\infty} + H^\infty$ then $\|H_\phi\| = \|\phi + \mathbf{H}_r^\infty\|$. However even if $\phi \in \mathscr{L}_r^\infty$ for $r = 0$ or $-\infty$, we do not know whether this equality holds. If $\phi \in \overline{\mathscr{H}_r^\infty} + \mathscr{H}_r^\infty$ then $\phi - \hat{\phi}(0,0) \in \overline{W\mathbf{H}_s^\infty}$ for some s with $-\infty \le s \le 0$. Hence $\|H_\phi\| = \|\phi + \mathbf{H}_s^\infty\|$.

Now we will give two corollaries about essential norms of Hankel operators and Toeplitz operators.

Corollary 1. *Put $r = 0$ or $-\infty$. If ϕ is in $\overline{W\mathbf{H}_r^\infty} + H^\infty$ then*

$$\|H_\phi\|_e = \|H_\phi\| = \|\phi + H^\infty\|.$$

Proof. By Theorem 2 and Lemma 2,

$$\|H_{Z^n\phi}\| = \|Z^n\phi + H^\infty\| = \|\phi + H^\infty\|$$

because $Z^n\phi \in \overline{W\mathbf{H}_r^\infty} + H^\infty$. Since $Z = z$ and $W = w$ for $r = 0$, and $Z = w$ and $W = z$ for $r = -\infty$, both T_Z and T_W are unilateral shift operators on H^2 with infinite multiplicities. By the proof for classical Hardy space (cf. [1]), the theorem follows. □

Corollary 2. *Let ϕ be a unimodular function in L^∞.*

(1) *For any fixed r with $-\infty \le r \le 0$, suppose ϕ is in $\overline{W\mathbf{H}_r^\infty} + H^\infty$. Then T_ϕ is left invertible if and only if $\|\phi + H^\infty\| < 1$.*

(2) *For $r = 0$ or $-\infty$, suppose ϕ is in $\mathscr{L}_r^\infty$. Then T_ϕ is left invertible if and only if $\|\phi + H^\infty\| < 1$.*

(3) *For $r = 0$ or $-\infty$, suppose ϕ is in $\overline{W\mathbf{H}_r^\infty}$. Then T_ϕ is left Fredholm if and only if $\|\phi + H^\infty\| < 1$.*

Proof. Using the well known equality

$$T_\phi^* T_\phi + H_\phi^* H_\phi = I$$

we can show (1) and (2) by Theorem 2 and Theorem 3, respectively, and (3) by Theorem 4. □

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