

THE PARAMETERS OF A CHAIN SEQUENCE

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ABSTRACT. We give a method for constructing explicitly all parameter sequences for any chain sequence for which one parameter sequence is known. An application to orthogonal polynomials associated with birth and death processes is given.

1. INTRODUCTION

A sequence $\{a_n\}_{n=1}^{\infty}$ is a (positive) chain sequence if there exists a second sequence $\{g_n\}_{n=0}^{\infty}$ such that

$$(1.1) \quad \begin{aligned} & \text{(i)} \quad 0 \leq g_0 < 1, \quad 0 < g_n < 1 \quad \text{for } n \geq 1; \\ & \text{(ii)} \quad a_n = (1 - g_{n-1})g_n \quad \text{for } n \geq 1. \end{aligned}$$

Chain sequences seem to have appeared first in certain continued fractions studied by E. B. Van Vleck and the theory was formalized by Wall (see [13] for this and references). See [7, 9] for some examples of more recent applications to continued fraction theory. Our interest in chain sequences stems from the useful role they play in the study of orthogonal polynomials and their zeros (see for example [1, 2, 3, 5, 6, 12]). This connection is not surprising in view of the close relation of orthogonal polynomials to continued fractions. In their original form, chain sequences were not restricted to be positive and for the more general case the above definition must be modified by changing the inequalities to weak inequalities. For the applications to orthogonal polynomials, only positive chain sequences are considered so we adopt the more restrictive definition above and will not use the adjective "positive".

The numbers g_n above are called parameters of the chain sequence and $\{g_n\}$ is a parameter sequence for $\{a_n\}$. Every chain sequence has a *minimal* parameter sequence $\{m_n\}$ uniquely determined by the condition $m_0 = 0$, and it has a *maximal* parameter sequence $\{M_n\}$ which is *characterised* by the condition

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[13, p. 82]

$$(1.2) \quad \sum_{n=1}^{\infty} \frac{M_1 M_2 \cdots M_n}{(1-M_1)(1-M_2) \cdots (1-M_n)} = \infty.$$

If $M_0 > 0$, then for every g_0 , $0 < g_0 < M_0$, there is a unique parameter sequence $\{g_n\}$ such that

$$(1.3) \quad m_n < g_n < M_n, \quad n \geq 0.$$

It is frequently useful to know explicitly more than one parameter sequence for a given chain sequence but it is rare that more than one or two are explicitly known. In some instances, a non-minimal parameter sequence is known but the minimal parameter sequence is what is needed. In this paper, we will show how *all* parameter sequences can be found for any chain sequence for which one parameter sequence is explicitly known.

2. NON-MINIMAL PARAMETERS KNOWN

We first introduce the following notation: Let $g = \{g_n\}$ be any parameter sequence and define $P_n = P_n(g)$ by

$$(2.1) \quad P_0(g) = 1, \quad P_n(g) = \frac{g_0 \cdots g_{n-1}}{(1-g_0) \cdots (1-g_{n-1})}, \quad n \geq 1.$$

Now suppose that (1.1) holds with $g_0 > 0$. Define the sequence $\{S_n\}$ and the extended real number S by

$$(2.2) \quad S_{-1} = 0, \quad S_n = \sum_{k=0}^n P_k \quad (n \geq 0), \quad S = \sum_{k=0}^{\infty} P_k.$$

Next define $h_n = h_n(t)$ by

$$(2.3) \quad h_n(t) = \frac{1 + tS_{n-1}}{1 + tS_n} g_n, \quad n \geq 0.$$

Since $S_n - g_n S_{n-1} = (1 - g_n)S_{n+1}$, we have

$$(2.4) \quad 1 - h_n(t) = \frac{1 + tS_{n+1}}{1 + tS_n} (1 - g_n).$$

Thus $(1 - h_{n-1})h_n = (1 - g_{n-1})g_n = a_n$ ($n \geq 1$) and it is readily verified that $0 \leq h_0 < 1$ and $0 < h_n < 1$ ($n \geq 1$) if and only if

$$(2.5) \quad -\frac{1}{S} \leq t \leq \infty.$$

(This includes the limiting case $S = \infty$.)

Thus with the restriction (2.5), (2.3) yields all parameter sequences $\{h_n\}$ for the chain sequence $\{a_n\}$. In particular, $h_0(\infty) = 0$ so we get the *minimal* parameters for $t = \infty$. On the other hand, we have

$$(2.6) \quad \frac{h_n}{1 - h_n} = \frac{1 + tS_{n-1}}{1 + tS_{n+1}} \cdot \frac{g_n}{1 - g_n},$$

$$(2.7) \quad P_n(h) = \frac{(1+t)(1+tS_1)}{(1+tS_n)(1+tS_{n+1})} P_n(g).$$

Recalling the criterion (1.2), we see we will have the *maximal* parameters if $t = -1/S$. This remains true if $S = \infty$ (in which case of course $h_n(0) = g_n$ is the n th maximal parameter).

Example. Take $a_n = a$ where $0 < a \leq 1/4$. Then $a_n = (1 - g)g$ with $g = [1 + \sqrt{1 - 4a}]/2$, $n \geq 1$. Referring to (2.1)–(2.3), we have

$$P_n = R^n, \quad R = \frac{1 + \sqrt{1 - 4a}}{1 - \sqrt{1 - 4a}}, \quad S_n = \frac{R^{n+1} - 1}{R - 1}.$$

Hence all parameter sequences for $\{a\}$ are given by

$$(2.8) \quad h_n(t) = \frac{1 + \sqrt{1 - 4a}}{2} \frac{R - 1 + (R^n - 1)t}{R - 1 + (R^{n+1} - 1)t}, \quad 0 \leq t \leq \infty.$$

The corresponding maximal parameters $M_n = h_n(0) = g$ and the minimal parameters $m_n = h_n(\infty)$ were given by Wall [13, p. 83].

In the special case $a = 1/4$, (2.8) reduces neatly to

$$(2.9) \quad h_n(t) = \frac{1 + nt}{2[1 + (n + 1)t]}.$$

3. MINIMAL PARAMETERS ARE KNOWN

Suppose next that

$$(3.1) \quad a_n = (1 - m_{n-1})m_n, \quad n \geq 1,$$

with $m_0 = 0$, $0 < m_n < 1$ ($n \geq 1$) and

$$(3.2) \quad \sigma = \sum_{k=1}^{\infty} \frac{m_1 \cdots m_n}{(1 - m_1) \cdots (1 - m_n)} < \infty.$$

(If the series in (3.2) diverges, the minimal parameters are also the maximal parameters so the chain sequence has only the one parameter sequence.)

Now define a second sequence $\{b_n\}$ by

$$(3.3) \quad b_n = a_{n+1}, \quad n \geq 1.$$

Then $b_n = (1 - g_{n-1})g_n$ where

$$(3.4) \quad g_n = m_{n+1}, \quad n \geq 0,$$

so $\{b_n\}$ is a chain sequence with the non-minimal parameter sequence $\{g_n\}$. Thus we can define $\{S_n\}$ and S by (2.2). In particular, we have $S = 1 + \sigma$, where σ is given by (3.2). For each t satisfying (2.5), (2.3) defines a parameter sequence for $\{b_n\}$. As before, $h_n(-1/S) (= M_{n+1})$ is the n th maximal parameter for $\{b_n\}$ while $h_n(0) = g_n = m_{n+1}$. Also, we have

$$(3.5) \quad a_{n+1} = b_n = (1 - h_{n-1})h_n, \quad n \geq 1,$$

where

$$m_{n+1} \leq h_n(t) \leq M_{n+1} \quad \text{for} \quad -\frac{1}{S} \leq t \leq 0.$$

Further, $h_0(t) = m_1/(1+t)$ so if we define h_{-1} by

$$(3.6) \quad h_{-1} = -t, \quad -\frac{1}{S} \leq t \leq 0,$$

then

$$(1 - h_{-1})h_0 = m_1 = a_1.$$

Combined with (3.5), this shows that $\{h_n(t)\}_{n=-1}^{\infty}$ is a parameter sequence for $\{a_n\}_{n=1}^{\infty}$. Pulling all of this together, we set $r = -t$, $f_n(r) = h_{n-1}(t)$, and we can state in summary:

Let $\{a_n\}_{n=1}^{\infty}$ be a chain sequence with *minimal* parameter sequence $\{m_n\}_{n=0}^{\infty}$ satisfying (3.2). Let

$$(3.7) \quad S_{-1} = 0, \quad S_0 = 1, \quad S_n = 1 + \sum_{k=1}^n \frac{m_1 \cdots m_k}{(1-m_1) \cdots (1-m_k)}, \quad n \geq 1$$

Then all parameter sequences for $\{a_n\}$ are given by $\{f_n(r)\}$ where

$$(3.8) \quad f_0(r) = r, \quad f_n(r) = \frac{1 - rS_{n-1}}{1 - rS_n} m_n, \quad 0 \leq r \leq \frac{1}{S}.$$

An example in which S_n can be found in closed form is given by taking

$$a_n = \frac{(2n-1)(2n+3)}{16n(n+1)} \quad \text{with } m_n = \frac{n^2(2n+3)}{4(n+1)^3}.$$

All parameter sequences are then given by (3.8) with $0 \leq r \leq 3/4$ and $S_n = (4/3)(n+1)(n+3)(n+2)^{-2}$.

4. APPLICATIONS

If $\{f_n\}$ is any sequence, put

$$(4.1) \quad \pi_n(f) = \frac{f_1 f_2 \cdots f_n}{(1-f_1)(1-f_2) \cdots (1-f_n)}.$$

Then if $\{g_n\}$ and $\{h_n\}$ are any two *non-maximal* parameter sequences for the same chain sequence, (2.6) shows that

$$(4.2) \quad \lim_{n \rightarrow \infty} \frac{\pi_n(g)}{\pi_n(h)} = \frac{(1+t)(1+tS_1)}{(1+tS)^2} > 0,$$

where S (given by (2.2)) is finite and $t = (g_0 - h_0)/h_0$. (We had previously known that $\lim_{n \rightarrow \infty} g_n/h_n = 1$.)

Convergence of certain series involving $\pi_n(m)$, where m is the *minimal* parameter sequence, figures prominently in the question of determinacy or indeterminacy of the Hamburger moment problems associated with orthogonal polynomials (see [5]). In many cases, we do not know the minimal parameter sequence but (4.2) shows that all non-maximal parameter sequences are asymptotically equivalent.

As a second application, consider the problem of explicitly constructing a family of orthogonal polynomial sequences which are orthogonal over $[0, \infty)$ with respect to measures which differ from each other only by the size of the mass at the origin. This construction was described in [1, Th. 2] and is especially well suited to the case where the orthogonal polynomials are associated with a birth and death process. Specifically, let $\{Q_n(x)\}$ be defined by the recurrence formula (see [8])

$$(4.3) \quad \begin{aligned} -xQ_n(x) &= \lambda_n Q_{n+1}(x) + (\lambda_n + \mu_n)Q_n(x) + \mu_n Q_{n-1}(x), \\ Q_{-1}(x) &= 0, \quad Q_0(x) = 1, \quad \lambda_i > 0, \quad \mu_{i+1} > 0 \quad (i \geq 0), \quad \mu_0 \geq 0. \end{aligned}$$

These polynomials satisfy an orthogonality of the form

$$\int_0^\infty Q_m(x)Q_n(x) d\psi(x) = \frac{\mu_1 \cdots \mu_n}{\lambda_0 \cdots \lambda_{n-1}} \delta_{mn}.$$

Let us consider here the case where we have a reflecting barrier at 0—that is, assume $\mu_0 = 0$. (If $\mu_0 > 0$, the procedure becomes a bit more complicated.)

We then consider the corresponding “dual process”. That is, consider the process obtained from (4.3) after replacing λ_n and μ_n , respectively, by λ_{n+1} and μ_{n+1} . The resulting orthogonal polynomials are the “kernel polynomials” which are orthogonal with respect to the distribution $x d\psi(x)$. Consideration of the corresponding *monic* polynomials and the resulting recurrence relation leads to the chain sequence

$$(4.4) \quad \frac{\lambda_n \mu_n}{(\lambda_{n-1} + \mu_n)(\lambda_n + \mu_{n+1})} = (1 - g_{n-1})g_n, \quad g_n = \frac{\lambda_n}{\lambda_n + \mu_{n+1}}.$$

Next obtain all other parameter sequences for (4.4). That is, put

$$\rho_n(t) = \frac{(1 + tS_{n-1})}{1 + tS_n},$$

where S_n is given by (2.2), and then define

$$\lambda_n(t) = \rho_n \lambda_n, \quad \mu_0(t) = 0, \quad \mu_{n+1}(t) = \frac{1}{\rho_n} \mu_{n+1}.$$

Now consider the orthogonal polynomials defined by (4.3) after replacing λ_n and μ_n by $\lambda_n(t)$ and $\mu_n(t)$, respectively. Then the following is true [1, Theorem 2]: for $t = -1/S$ (corresponding to the maximal parameters of (4.4)), the corresponding polynomials will be orthogonal over $[0, \infty)$ with respect to a distribution $d\varphi(x)$ where φ is the solution of a *determined* Hamburger moment problem and is continuous at the origin. For all other finite values of t (corresponding to all other non-minimal parameter sequences), the corresponding polynomials will be orthogonal with respect to the distribution which is obtained from $d\varphi(x)$ by adding positive mass J at the origin, the mass being given by the formula

$$(4.5) \quad J = (1 + t) \frac{S}{S - 1} [\varphi(\infty) - \varphi(0^-)].$$

Note that the moment problem associated with the original orthogonal polynomial sequence $\{Q_n(x)\}$ could be indeterminate. Thus if a polynomial sequence is orthogonal over $[0, \infty)$ with respect to the solution of an indeterminate moment problem, it is also orthogonal with respect to a distribution which is obtained by adding mass at the origin to a solution of a determined Hamburger moment problem.

The above is a more efficient and generally applicable method of determining this family of orthogonal polynomials than that used in [4]. For other approaches to the problem of determining orthogonal polynomials when masses are added to measures under various conditions, see [10, 11].

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