PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 108, Number 3, March 1990

A NOTE ON THE NUMBER OF PRIMES IN SHORT INTERVALS

D. A. GOLDSTON AND S. M. GONEK

(Communicated by William Adams)

ABSTRACT. Let $J(\beta, T) = \int_1^{T^\beta} (\sum_{x<p^k\le x+x/T} \log p - x/T)^2\,dx/x^2$, where the sum is over prime powers. H. L. Montgomery has shown that on the Riemann hypothesis, there is a positive constant C_0 such that for each $\beta \ge 1$, $J(\beta, T) \le C_0\beta \log^2 T/T$, provided that T is sufficiently large. Here we prove a slightly stronger result from which we deduce a lower bound of the same order.

1. INTRODUCTION

In 1943 A. Selberg [7] proved that if the Riemann hypothesis (RH) is true, then

$$J(\beta, T) = \int_1^{T^\beta} \left(\psi\left(x+\frac{x}{T}\right) - \psi(x) - \frac{x}{T}\right)^2 x^{-2}\,dx \ll_\beta \frac{\log^2 T}{T}$$

for fixed $\beta \ge 1$ and $T \ge 2$; here $\psi(x) = \sum_{n\le x}\Lambda(n)$, where $\Lambda(n) = \log p$ if $n = p^m$ with p a prime number and $m \ge 1$, and $\Lambda(n) = 0$ otherwise. H. L. Montgomery (unpublished) later made the β dependence explicit by proving that on RH there exists an absolute constant C_0 such that, for each $\beta \ge 1$,

$$J(\beta, T) \le C_0 \frac{\beta \log^2 T}{T} \tag{1}$$

as $T \to \infty$. Proofs of this subsequently appeared in [1], [5], and [4]. Our object here is to prove a stronger result for $J(\beta, T)$ on RH which immediately implies (1) and, moreover, shows that apart from constants (1) is best possible.

We shall use the standard symbols $\ll$, $\gg$ O, o, and $\sim$ and, unless otherwise indicated, all implied constants will be absolute.

Theorem. *Assume the Riemann hypothesis. Then there are absolute constants* $C_2 > C_1 > 0$ *such that for each* $\beta > 0$,

$$C_1 \frac{\log^2 T}{T} \le J(\beta + 2, T) - J(\beta, T) \le C_2 \frac{\log^2 T}{T}$$

for all sufficiently large T.

Received by the editors March 13, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 11M26, 11N05.
Research of both authors was partially supported by grants from the National Science Foundation.

Corollary. *Assume the Riemann hypothesis. Then there are absolute constants* $D_2 > D_1 > 0$ *such that, for each* $\beta \geq 1$,

$$D_1 \frac{\beta \log^2 T}{T} \leq J(\beta, T) \leq D_2 \frac{\beta \log^2 T}{T}$$

for all sufficiently large T.

The Theorem should be compared with a result of Gallagher and Mueller [1] (also see [3]) which asserts that RH and the pair correlation conjecture together imply that for fixed $\beta_1 > \beta_0 \geq 1$,

$$J(\beta_1, T) - J(\beta_0, T) = ((\beta_1 - \beta_0) + o(1)) \frac{\log^2 T}{T} \qquad (\text{as } T \to \infty).$$

Since for $0 < \beta \leq 1$ one also has (unconditionally) that

$$J(\beta, T) \sim \frac{\beta^2}{2} \frac{\log^2 T}{T} \qquad (\text{as } T \to \infty) \tag{2}$$

(see [1]), we see that on the above hypotheses

$$J(\beta + 2, T) - J(\beta, T) \sim \begin{cases} (3/2 + \beta - \beta^2/2)\frac{\log^2 T}{T} & \text{if } 0 < \beta \leq 1, \\ 2\frac{\log^2 T}{T} & \text{if } \beta \geq 1. \end{cases}$$

Our proof will actually show that if $\beta > 0$, then

$$.3 \frac{\log^2 T}{T} \leq J(\beta + 2, T) - J(\beta, T) \leq 21.65 \frac{\log^2 T}{T}$$

for all sufficiently large T. It is also possible by our method to show that

$$(\beta_1 - \beta_0) \frac{\log^2 T}{T} \ll J(\beta_1, T) - J(\beta_0, T) \ll (\beta_1 - \beta_0) \frac{\log^2 T}{T}$$

for $\beta_1 > \beta_0 > 0$ as long as $\beta_1 - \beta_0 > 6 - 2\sqrt{6} = 1.10102\ldots$. It is doubtful, however, whether one can obtain this for arbitrarily small differences $\beta_1 - \beta_0$ on RH alone.

2. A lemma

We prove the Theorem by relating $J(\beta, T)$ to averages of the function

$$F(\alpha, T) = \left(\frac{T}{2\pi} \log T \right)^{-1} \sum_{0 < \gamma, \gamma' \leq T} T^{i\alpha(\gamma - \gamma')} w(\gamma - \gamma')$$

introduced by Montgomery [6]; here α is real, $T \geq 2$, $w(u) = 4/4 + u^2$, and γ, γ' denote the imaginary parts of zeros of the Riemann zeta-function. We shall then require the following result which generalizes and strengthens Lemma A of [2].

Lemma. *Assume the Riemann hypothesis and let*

$$G(\alpha, T) = \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{0<\gamma,\gamma'\le T} \left(\frac{\sin\frac{\alpha}{2}(\gamma-\gamma')\log T}{\frac{\alpha}{2}(\gamma-\gamma')\log T}\right)^2 w(\gamma-\gamma').$$

Then for $a > 0$, β *real, and* $T \ge 2$,

$$a\left(1-\tfrac{1}{2}G\left(\tfrac{a}{2}, T\right)\right) \le \int_\beta^{\beta+a} F(\alpha, T)\,d\alpha \le a\left(G(a, T) + \tfrac{1}{2}G\left(\tfrac{a}{2}, T\right)\right). \tag{3}$$

Proof. The proof of the lower bound in Lemma A (which corresponds to $a = 2$ here) extends in a straightforward way to give the lower bound in (3).

On the other hand, the upper bound in Lemma A generalizes to $2aG(a, T)$ which is not as good as the bound in (3).

To obtain the present upper bound define $K_b(u) = \max(1-|u|/b, 0)$, $b > 0$, and consider the function

$$R_a(u) = K_a(u) + \frac{1}{2}K_{a/2}(u-a/2) + \frac{1}{2}K_{a/2}(u+a/2).$$

Defining the Fourier transform of $f(x)$ by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e(-x\omega)\,dx,$$

where $e(u) = e^{2\pi i u}$, we have that

$$\widehat{K}_b(\omega) = b\left(\frac{\sin \pi b\omega}{\pi b\omega}\right)^2.$$

Thus

$$\widehat{R}_a(\omega) = a\left\{\left(\frac{\sin\pi a\omega}{\pi a\omega}\right)^2 + \frac{1}{2}\cos\pi a\omega\left(\frac{\sin\pi a\omega/2}{\pi a\omega/2}\right)^2\right\}.$$

Now clearly $R_a(u) \ge 0$ for all u, and $R_a(u) = 1$ for $|u| \le a/2$. Furthermore (see [5]), $F(\alpha, T) \ge 0$. Hence

$$\begin{aligned}
\int_\beta^{\beta+a} F(\alpha, T)\,d\alpha &= \int_{(\beta+a/2)-a/2}^{(\beta+a/2)+a/2} F(\alpha, T)\,d\alpha \\
&\le \int_{-\infty}^{\infty} F(\alpha, T)R_a(\alpha-(\beta+a/2))\,d\alpha \\
&= \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{0<\gamma,\gamma'\le T} T^{i(\beta+a/2)(\gamma-\gamma')}\widehat{R}_a\left(\frac{(\gamma-\gamma')\log T}{2\pi}\right) \\
&\quad \times w(\gamma-\gamma') \\
&\le a\left(G(a, T) + \frac{1}{2}G(a/2, T)\right).
\end{aligned}$$

This proves the result.

We require the lower bound of the lemma for $a = 2-\eta$ and the upper bound for $a = 2+\eta$, where $\eta > 0$. Montgomery [6] has shown that on RH,

$$G(\alpha, T) \sim \left(\frac{1}{\alpha} + \frac{\alpha}{3}\right)$$

for $0 < \alpha \leq 1$ as $T \to \infty$. Hence

$$(4) \qquad \int_{\beta}^{\beta+2-\eta} F(\alpha, T)\, d\alpha \geq (1+o(1))\left(\frac{2}{3} - c_1(\eta)\right) \qquad (\text{as } T \to \infty),$$

where $c_1(\eta) \to 0$ as $\eta \to 0^+$. For the upper bound we use the inequality

$$G(j+\eta, T) \leq 4/3 + c_2(\eta) + o(1)$$

as $T \to \infty$ ($j = 1$ or 2), where $c_2(\eta) \to 0$ as $\eta \to 0^+$; these follow from Lemma 7 of [2]. We then obtain

$$(5) \qquad \int_{\beta}^{\beta+2+\eta} F(\alpha, T)\, d\alpha \leq (1+o(1))(4 + c_3(\eta)) \qquad (\text{as } T \to \infty),$$

where $c_3(\eta) \to 0$ as $\eta \to 0^+$.

3. Proof of the theorem and corollary

We begin by quoting two results from Goldston [3]. We remind the reader that the Riemann hypothesis is assumed throughout this section.

Let $g(x)$ be a complex valued function such that $g(x) \ll (1+x^2)^{-1}$, $\hat{g}(\omega) \ll (1+\omega^2)^{-1}$, and $\hat{g}(\omega) = 0$ for $\omega \leq 0$, and define

$$(6) \qquad H_{\pm}(\mu, U) = \int_0^U \left|\sum_{\gamma} g(\pm(t-\gamma)\mu)\right|^2 dt,$$

where the sum is over the ordinates of the nontrivial zeros of $\zeta(s)$. Then by Equations (5.2) and (5.3) of [3],

$$(7) \qquad H_{\pm}(\mu, U) = U \int_1^{\infty} F(\alpha, U) |\hat{g}(\alpha)|^2\, d\alpha + o(U)$$

for $|\mu - 1/2\pi \log U| \leq C \log\log U$, where C is an arbitrary positive constant. Furthermore, by Lemma 5 of [3] [1]

$$\int_0^{\infty} (\psi(e^{u+\delta}) - \psi(e^u) - (e^{\delta}-1)e^u)^2 e^{-u} \left|\hat{g}\left(\frac{u}{2\pi\nu}\right)\right|^2 du$$
$$= 8\pi\nu^2 \int_{-\infty}^{\infty} \left(\frac{\sin \frac{\delta}{2} t}{t}\right)^2 \left|\sum_{\gamma} g((t-\gamma)\nu)\right|^2 dt$$
$$+ O(\delta) + O(\nu^2 \delta^{3/2} \log 1/\delta)$$

[1] Lemma 5 contains two typographical errors: "$T \geq 2$" should be removed and the factor $(\psi(e^{u+\delta}) - \psi(e^u) - (e^{\delta}-1)e^u)$ in (4.2) should be squared.

uniformly for $\nu \geq 1$ and $0 < \delta \leq 1/4$. If we set $e^{\delta} = 1+1/T$, $\nu = 1/2\pi \log T$, and $x = e^{u}$, we may rewrite this as

(8)

$$\int_1^\infty \left(\psi\left(x+\frac{x}{T}\right) - \psi(x) - \frac{x}{T}\right)^2 \left|\hat{g}\left(\frac{\log x}{\log T}\right)\right|^2 \frac{dx}{x^2}$$
$$= \frac{2}{\pi}\log^2 T \int_0^\infty \left(\frac{\sin\frac{\delta}{2}t}{t}\right)^2 \left(\left|\sum_\gamma g\left((t-\gamma)\frac{\log T}{2\pi}\right)\right|^2 + \left|\sum_\gamma g\left((\gamma-t)\frac{\log T}{2\pi}\right)\right|^2\right) dt + O\left(\frac{1}{T}\right).$$

We now choose a pair of functions g, $\hat{g}$ for which the above conditions hold and such that $\hat{g}$ approximates the characteristic function of the interval $[\beta, \beta+b]$ from below, with $\beta > 0$. More specifically, we take $\hat{g} = 0$ off of $[\beta, \beta+b]$, $\hat{g} = 1$ on $[\beta+\eta, \beta+b-\eta]$, and $|\hat{g}| \leq 1$ otherwise, where $0 < \eta < b/2$ is fixed. (Such a pair may be constructed explicitly by a linear change of variable from the pair defined in (3.6) of [3].) With this choice of $\hat{g}$ in (7) we immediately obtain

$$(9)\quad U\int_{\beta+\eta}^{\beta+b-\eta} F(\alpha, U)\,d\alpha + o(U) \leq H_{\pm}(\mu, U) \leq U\int_\beta^{\beta+b} F(\alpha, U)\,d\alpha + o(U)$$

for $|\mu - 1/2\pi \log U| \leq C \log\log U$. The same choice in the left-hand side of (8) leads to the inequalities

$$(10)\quad \begin{aligned} &J(\beta+b-\eta, T) - J(\beta+\eta, T) \\ &\quad\leq \int_1^\infty \left(\psi\left(\frac{x}{T}\right) - \psi(x) - \frac{x}{T}\right)^2 \left|\hat{g}\left(\frac{\log x}{\log T}\right)\right|^2 \frac{dx}{x^2} \\ &\quad\leq J(\beta+b, T) - J(\beta, T). \end{aligned}$$

We now obtain the lower bound of the theorem. Taking $b = 2$ in (10) and using (8), we see that

$$J(\beta+2, T) - J(\beta, T)$$
$$\geq \frac{2}{\pi}\log^2 T \int_0^{\theta T} \left(\frac{\sin\frac{\delta}{2}t}{t}\right)^2 \left\{\left|\sum_\gamma g\left((t-\gamma)\frac{\log T}{2\pi}\right)\right|^2 + \left|\sum_\gamma g\left((\gamma-t)\frac{\log T}{2\pi}\right)\right|^2\right\} dt + O\left(\frac{1}{T}\right),$$

where $0 < \theta < \pi$. Now $(\sin(\delta t/2)/t)^2$ is monotone decreasing for $0 \leq t \leq \theta T$, so by (6) this is

$$\geq \frac{2}{\pi}\log^2 T \left(\frac{\sin\frac{\delta}{2}\theta T}{\theta T}\right)^2 \left\{H_+\left(\frac{\log T}{2\pi}, \theta T\right) + H_-\left(\frac{\log T}{2\pi}\theta T\right)\right\} + O\left(\frac{1}{T}\right).$$

Next, using the lower bound in (9), we find that this is

$$\geq \frac{4}{\pi}\frac{(\sin\delta\theta T/2)^2}{\theta}\frac{\log^2 T}{T}\int_{\beta+\eta}^{\beta+2-\eta} F(\alpha,\theta T)\,d\alpha + o\left(\frac{\log^2 T}{T}\right).$$

Finally, by (4) we have that

$$J(\beta+2,T) - J(\beta,T) \geq \frac{4}{\pi}\frac{(\sin\theta/2)^2}{\theta}\frac{\log^2 T}{T}(2/3 - c_1(\eta)) + o\left(\frac{\log^2 T}{T}\right).$$

The optimal choice of θ is the unique solution (on $(0,\pi)$) of the equation $\tan\theta/2 = \theta$, namely $\theta = 2.33112\ldots$. Using this and taking η sufficiently small, we obtain

$$\begin{aligned} J(\beta+2,T) - J(\beta,T) &\geq (.307 + o(1))\frac{\log^2 T}{T} \\ &\geq .3\frac{\log^2 T}{T}, \end{aligned}$$

for $\beta > 0$ and all T sufficiently large.

To obtain the upper bound we again take g and $\hat{g}$ as above (although b will be different). By the growth condition on g and the estimate $\sum_{u-1<\gamma\leq u} 1 \ll \log(|u|+2)$, we easily obtain the bound

$$\left|\sum_\gamma g\left(\pm(t-\gamma)\frac{\log T}{2\pi}\right)\right| \ll \log(|t|+2).$$

Using this, we find that

$$\begin{aligned} &\int_0^\infty \left(\frac{\sin\frac{\delta}{2}t}{t}\right)^2 \left|\sum_\gamma g\left(\pm(t-\gamma)\frac{\log T}{2\pi}\right)\right|^2 dt \\ &\quad= \int_0^{T\log^3 T} \left(\frac{\sin\frac{\delta}{2}t}{t}\right)^2 \left|\sum_\gamma g\left(\pm(t-\gamma)\frac{\log T}{2\pi}\right)\right|^2 dt \\ &\qquad + O\left(\int_{T\log^3 T}^\infty \frac{\log^2 t}{t^2}\,dt\right) \\ &\quad\leq \frac{\delta^2}{4}\int_0^T \left|\sum_\gamma g\left(\pm(t-\gamma)\frac{\log T}{2\pi}\right)\right|^2 dt \\ &\qquad + \sum_{k=1}^{[5\log\log T]} \int_{2^{k-1}T}^{2^k T} \left|\sum_\gamma g\left(\pm(t-\gamma)\frac{\log T}{2\pi}\right)\right|^2 t^{-2}\,dt + o\left(\frac{1}{T}\right) \\ &\quad\leq \frac{\delta^2}{4}H_\pm\left(\frac{\log T}{2\pi}, T\right) + T^{-2}\sum_{k=1}^{[5\log\log T]} 2^{2-2k} H_\pm\left(\frac{\log T}{2\pi}, 2^k T\right) \\ &\qquad + o\left(\frac{1}{T}\right). \end{aligned}$$

The bound for $H_\pm$ from (9) is applicable in this range of k and leads to

$$\leq (1+o(1))\left\{\frac{1}{4T}\int_\beta^{\beta+b} F(\alpha, T)\,d\alpha + \frac{4}{T}\sum_{k=1}^{[5\log\log T]} 2^{-k}\int_\beta^{\beta+b} F(\alpha, 2^k T)\,d\alpha\right\}$$
$$+o\left(\frac{1}{T}\right).$$

If we now set $b = 2+2\eta$ and use (5) we find that this is

$$\leq (1+o(1))\left\{\frac{4+c_3(2\eta)}{4T} + \frac{4(4+c_3(2\eta))}{T}\right\} + o\left(\frac{1}{T}\right)$$
$$\leq \frac{17+5c_3(2\eta)+o(1)}{T}$$

as $T\to\infty$. Thus, we have shown that

$$\text{(11)} \qquad \int_0^\infty \left(\frac{\sin\frac{\delta}{2}t}{t}\right)^2 \left|\sum_\gamma g\left(\pm(t-\gamma)\frac{\log T}{2\pi}\right)\right|^2 dt \leq \frac{17+5c_3(2\eta)+o(1)}{T}$$

as $T\to\infty$.

We now combine (8), (10) (with $b = 2+2\eta$), and (11) to obtain

$$\begin{aligned} J(\beta+2+\eta, T) &- J(\beta+\eta, T) \\ &\leq \frac{2}{\pi}\log^2 T\int_0^\infty \left(\frac{\sin\frac{\delta}{2}t}{t}\right)^2 \\ &\quad\times\left\{\left|\sum_\gamma g\left((t-\gamma)\frac{\log T}{2\pi}\right)\right|^2 + \left|\sum_\gamma g\left((\gamma-t)\frac{\log T}{2\pi}\right)\right|^2\right\} dt \\ &\quad + O\left(\frac{1}{T}\right) \\ &\leq \frac{4}{\pi}(17+5c_3(2\eta)+o(1))\frac{\log^2 T}{T} \\ &\leq (21.646+7c_3(2\eta)+o(1))\frac{\log^2 T}{T} \end{aligned}$$

as $T\to\infty$. Taking η sufficiently small, we see that

$$J(\beta+2+\eta, T) - J(\beta+\eta, T) \leq 21.65\frac{\log^2 T}{T}$$

for $\beta > 0$ and all sufficiently large T. This gives the upper bound and completes the proof of the theorem.

We now prove the corollary. For $\beta > 2$ the corollary follows immediately from the theorem. Suppose then that $1 \leq \beta \leq 2$. By (2) and the fact that $J(\beta, T)$ is an increasing function of β we have

$$\frac{1}{2}\frac{\log^2 T}{T} \sim J(1, T) \leq J(\beta, T) \leq J(3, T).$$

Thus,

$$J(\beta, T) \gg \frac{\log^2 T}{T} \gg \beta \frac{\log^2 T}{T}$$

and, by the theorem,

$$\begin{aligned} J(\beta, T) &\le (J(3, T) - J(1, T)) + J(1, T) \\ &\ll \frac{\log^2 T}{T} \ll \beta \frac{\log^2 T}{T}. \end{aligned}$$

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Department of Mathematics and Computer Science, San Jose State University, San Jose, California 95192

Department of Mathematics, University of Rochester, Rochester, New York 14627