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ASYMPTOTIC DEPTH AND CONNECTEDNESS IN PROJECTIVE SCHEMES

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ABSTRACT. Let $I \subseteq \mathfrak{m}$ be an ideal of a local noetherian ring $(R, \mathfrak{m})$. Consider the exceptional fiber $\pi^{-1}(V(I))$ of the blowing-up morphism

$$\pi\colon \operatorname{Proj}\left(\bigoplus_{n\geq 0} I^n\right) \to \operatorname{Spec}(R)$$

and the special fiber $\pi^{-1}(\mathfrak{m})$. We show that the complement set

$$\pi^{-1}(V(I)) - \pi^{-1}(\mathfrak{m})$$

is highly connected if the asymptotic depth of the higher conormal modules I^n/I^{n+1} is large.

1. INTRODUCTION

Let $(R, \mathfrak{m})$ be a local noetherian ring and let $I \subseteq R$ be an ideal of height > 0. Then it is known that the depths of the R-modules R/I^n and I^n/I^{n+1} take constant values $t(I)$ resp. $\bar{t}(I)$ for all large n [2]:

$$\text{(1.1)} \qquad \begin{aligned} &\text{(i) } \operatorname{depth}(R/I^n) = t(I) \quad \forall n \gg 0, \\ &\text{(ii) } \operatorname{depth}(I^n/I^{n+1}) = \bar{t}(I) \quad \forall n \gg 0. \end{aligned}$$

$t(I)$ and $\bar{t}(I)$ are called the *asymptotic depths* of R/I^n resp. of I^n/I^{n+1}. In [3] we have shown

$$\text{(1.2)} \qquad \bar{t}(I) \geq t(I).$$

It turns out that these asymptotic depths are related to the topology of the blowing-up of $\operatorname{Spec}(R)$ at I, which by definition is given by the canonical morphism

$$\text{(1.3)} \qquad \operatorname{Bl}(I) := \operatorname{Proj}(\mathfrak{R}(I)) \xrightarrow{\pi_I} \operatorname{Spec}(R),$$

where $\mathfrak{R}(I)$ stands for the Rees algebra $\bigoplus_{n\geq 0} I^n$ of I. It was noticed by Burch [7] that the dimension of the special fiber $\pi_I^{-1}(\mathfrak{m})$ of (1.3) is subject to the inequality $\dim(\pi_I^{-1}(\mathfrak{m})) < \dim(R) - \min_n \operatorname{depth}(R/I^n)$.

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In [2] we gave an improvement of this, by showing that $\min_n \operatorname{depth}(R/I^n)$ may be replaced by the asymptotic depth $t(I)$ of the rings R/I^n. In fact, we even may replace $\min_n \operatorname{depth}(R/I^n)$ by $\bar{t}(I)$ (cf. [3]):

(1.4) $$\dim(\pi_I^{-1}(\mathfrak{m})) < \dim(R) - \bar{t}(I).$$

So, if the asymptotic depth $\bar{t}(I)$ is large, the special fiber $\pi_I^{-1}(\mathfrak{m})$ must be small.

In this note we want to give a further result of this type. We namely shall prove that for large values of $\bar{t}(I)$ the complement

(1.5) $$\pi_I^{-1}(V(I)) - \pi_I^{-1}(\mathfrak{m})$$

of the special fiber in the exceptional fiber $\pi_I^{-1}(V(I))$ is highly connected under certain additional conditions (4.8).

In fact we shall give our result in a more general context. Instead of the blowing-up morphism (1.3) we consider an arbitrary projective morphism π: $\operatorname{Proj}(S) \to \operatorname{Spec}(R)$ (induced by a homogeneous R-algebra S). We then choose a noetherian graded S-module $M = \bigoplus_{n\in\mathbb{Z}} M_n$ and relate the depths of the R-modules M_n to the connectivity of the sheaf $\mathscr{F}$ which is induced by M on $\operatorname{Proj}(S)$. Using the connectedness criteria for blowing-up given in [5], we immediately will obtain bounds of connectivity for the set (1.5).

2. Asymptotic depth in projective schemes

Throughout this section let $(R, \mathfrak{m})$ be a local noetherian ring and let $S = R \oplus S_1 \oplus S_2 \oplus \cdots$ ($S_n \neq 0\ \forall n \gg 0$) be a homogeneous noetherian R-algebra. So we may write $S = R[a_1, \dots, a_r]$ with $a_1, \dots, a_r \in S_1$. We consider the canonical morphisms

(2.1) $$\gamma\colon \operatorname{Spec}(S) \to \operatorname{Spec}(R), \qquad \pi\colon \operatorname{Proj}(S) \to \operatorname{Spec}(R).$$

Moreover, let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ ($M_n \neq 0\ \forall n \gg 0$) be a finitely generated, graded essential S-module.

Let $\mathscr{F}$ be the coherent sheaf ($\neq 0$) induced by M on $\operatorname{Proj}(S)$

(2.2) $$\mathscr{F} := \tilde{M}.$$

Finally we introduce

(2.3) $$\operatorname{depth}_*(M) := \min\{\operatorname{depth}(M_n) \mid n \in \mathbb{Z}\}.$$

(We make use of the convention depth $(0) = \infty$.) Denoting the *grade* of M with respect to an ideal $I \subseteq S$ by $g(I, M)$ (by definition this is the maximal length of M-regular sequences in I), we also may write

(2.3)′ $$\operatorname{depth}_*(M) = g(\mathfrak{m}S, M).$$

This is easily seen in expressing depth and grade by the vanishing of local cohomology and observing the natural isomorphisms $H^i_{\mathfrak{m}S}(M) \cong \bigoplus_{n\in\mathbb{Z}} H^i_{\mathfrak{m}}(M)$.

(2.4) **Proposition.** (i) $\bigcup_{n\in\mathbb{Z}} \mathrm{Ass}(M_n) = \gamma(\mathrm{Ass}(M))$.

(ii) $\mathrm{depth}_*(M) = \min\{\mathrm{depth}(M_\mathfrak{p}) | \mathfrak{p} \in \gamma^{-1}(\mathfrak{m})\}$.

Proof. (i) After an eventual localization it suffices to show that $\mathfrak{m}$ belongs to $\mathrm{Ass}(M_n)$ for some n iff it belongs to $\gamma(\mathrm{Ass}(M))$. The first statement is equivalent to $\mathrm{depth}_*(M) = 0$, whereas the second one means $g(\mathfrak{m}S, M) = 0$. This allows to conclude by $(2.3)'$.

(ii) The right-hand side of the stated equality is just $g(\mathfrak{m}S, M)$. So we conclude again by $(2.3)'$.

(2.5) **Proposition.** *For all sufficiently large* $n \in \mathbb{N}$, *the following statements are true*:

(i) $\mathrm{Ass}(M_n) = \pi(\mathrm{Ass}(\mathscr{F}))$.

(ii) $\mathrm{depth}(M_n) = \min\{\mathrm{depth}(\mathscr{F}_x) | x \in \pi^{-1}(\mathfrak{m})\}$.

Proof. (i) After an eventual extension of R we may assume that $R/\mathfrak{m}$ is infinite. If we consider the points of $\mathrm{Proj}(S)$ as essential prime ideals in S, $\mathrm{Ass}(\mathscr{F})$ is exactly the set of essential members of $\mathrm{Ass}(M)$. So there is an $n_0 \in \mathbb{Z}$ with $\mathrm{Ass}(\mathscr{F}) = \mathrm{Ass}(M_{\geq n_0})$, where $M_{\geq n_0}$ stands for the submodule $\bigoplus_{n\geq n_0} M_n$ of M. Thus, replacing M by $M_{\geq n_0}$, we may write $\mathrm{Ass}(\mathscr{F}) = \mathrm{Ass}(M)$, hence $\pi(\mathrm{Ass}(\mathscr{F})) = \gamma(\mathrm{Ass}(M))$. Now, by our choice of M, $\mathrm{Ass}(M)$ has only essential members. So none of the (finitely many) $\mathfrak{p} \in \mathrm{Ass}(M)$ contains S_1. As $R/\mathfrak{m}$ is infinite this allows to choose an element $f \in S_1$ which avoids all members of $\mathrm{Ass}(M)$. We thus obtain injections $M_n \xrightarrow{f} M_{n+1}$, which show that $\mathrm{Ass}(M_n) \subseteq \mathrm{Ass}(M_{n+1})$, $\forall n \in \mathbb{Z}$. Now we conclude by (2.4)(i).

(ii) We make induction on $d_M = \min\{\mathrm{depth}(\mathscr{F}_x) | x \in \pi^{-1}(\mathfrak{m})\}$. If $d_M = 0$, we conclude by (i). If $d_M > 0$, clearly $\mathfrak{m} \notin \pi(\mathrm{Ass}(\mathscr{F}))$. We thus find an element $a \in \mathfrak{m}$ which avoids all members of $\pi(\mathrm{Ass}(\mathscr{F}))$. By (i), a becomes M_n-regular for all $n \gg 0$. So for all sufficiently large n we have $\mathrm{depth}(M_n/aM_n) = \mathrm{depth}(M_n) - 1$. Moreover by our choice of a, we have $\mathrm{depth}(\mathscr{F}_x/a\mathscr{F}_x) = \mathrm{depth}(\mathscr{F}_x) - 1$, thus $d_{M/aM} = d_M - 1$. So we conclude applying the hypothesis of induction to M/aM.

We introduce the following notations:

$$\text{(2.6)}\qquad \begin{aligned} &\text{(i) } \mathrm{Ass}^*(M) := \pi(\mathrm{Ass}(\mathscr{F})) = \mathrm{Ass}(M_n) \quad \forall n \gg 0, \\ &\text{(ii) } \mathrm{depth}^*(M) := \min\{\mathrm{depth}(\mathscr{F}_x) | x \in \pi^{-1}(\mathfrak{m})\} = \mathrm{depth}(M_n) \quad \forall n \gg 0. \end{aligned}$$

$\mathrm{Ass}^*(M)$ is called the *asymptotic set of prime divisors of* M_n whereas $\mathrm{depth}^*(M)$ is called the *asymptotic depth of* M_n. We notice

$$\text{(2.7)}\qquad \begin{aligned} &\text{(i) } \mathrm{Ass}^*(M) = \gamma(\mathrm{Ass}(M_{\geq n})) \quad \forall n \gg 0, \\ &\text{(ii) } \mathrm{depth}^*(M) = \mathrm{depth}_*(M_{\geq n}) \quad \forall n \gg 0. \end{aligned}$$

As an application of our previous results we want to prove

(2.8) **Corollary.** $\dim(\mathrm{Supp}(\mathscr{F}) \cap \pi^{-1}(\mathfrak{m})) \le \dim(\mathrm{Proj}(S)) - \mathrm{depth}^*(M)$.

Proof. Let x be a generic point of $\mathrm{Supp}(\mathscr{F}) \cap \pi^{-1}(\mathfrak{m})$. By (2.5)(ii) we have $\mathrm{depth}(\mathscr{F}_x) \ge \mathrm{depth}^*(M)$; thus $\dim(\mathscr{O}_{\mathrm{Proj}(S),x}) \ge \mathrm{depth}^*(M)$. This induces

$$\dim(\overline{\{x\}}) \le \dim(\mathrm{Proj}(S)) - \dim(\mathscr{O}_{\mathrm{Proj}(S),x}) \le \dim(\mathrm{Proj}(S)) - \mathrm{depth}^*(M),$$

hence our claim.

(2.9) *Remark.* (i) Let $(R, \mathfrak{m})$ be noetherian and local and let $I \subseteq \mathfrak{m}$ be an ideal of positive height. Let $\mathrm{Gr}(I) = \bigoplus_{n\ge 0} I^n/I^{n+1}$ be the associated graded ring of I. Applying (2.5)(ii) with $S = M = \mathrm{Gr}(I)$, we obtain the asymptotic stability of $\mathrm{depth}(I^n/I^{n+1})$ for $n \gg 0$ (cf. [2, 3]).

(ii) Let $(R, \mathfrak{m})$ and $I \subseteq R$ as in (i). Consider the canonical diagram

$$\begin{array}{ccc} \mathrm{Bl}_R(I) = \mathrm{Proj}(R(I)) & \xrightarrow{\pi_I} & \mathrm{Spec}(R) \\ \uparrow & & \uparrow \\ \pi^{-1}(V(I)) = \mathrm{Proj}(\mathrm{Gr}(I)) & \xrightarrow{\overline{\pi}_I} & \mathrm{Spec}(R/I) \end{array}$$

Then, applying (2.8) to $S = M = \mathrm{Gr}(I)$ and observing that $\dim(\mathrm{Proj}(\mathrm{Gr}(I))) = \dim(R) - 1$ we get $\dim(\pi_I^{-1}(\mathfrak{m})) = \dim(\overline{\pi}_I^{-1}(\mathfrak{m}/I)) \le \dim(R) - 1 - \overline{t}(I)$. This proves again (1.4). In the same way we get the asymptotic stability of the sets $\mathrm{Ass}(I^n/I^{n+1})$ which is shown in [1].

3. Connectedness-subdimension

We define the *subdimension* $\underline{\dim}(Z)$ of a closed set Z in a noetherian scheme X as the minimal codimension (in Z) of all closed points $x \in Z$. Thus we may write

$$\underline{\dim}(Z) = \min\{\dim_x(Z) \mid x \in Z \text{ closed }\} \qquad (\le \dim(Z)).$$

To be complete we define $\dim(\emptyset) = \underline{\dim}(\emptyset) = -1$.

If $Z \subseteq X$ is a closed subset, we define the *connectedness-subdimension* of Z as follows:

$$(3.1) \qquad \underline{c}(Z) := \min\{\underline{\dim}(W) \mid W \subseteq Z \text{ closed}, \ Z - W \text{ disconnected}\}.$$

Thereby the empty set $\emptyset$ is considered as disconnected. So we have $\underline{c}(Z) \ge -1$ with $\underline{c}(Z) \ge 0$ iff Z is connected. Comparing with the *connectedness-dimension* $c(Z)$ introduced as in [6], we obviously have $\underline{c}(Z) \le c(Z)$. If $Z \ne \emptyset$ we have

$$(3.2) \qquad \underline{c}(Z) = \min\{\underline{\dim}(Z_1 \cap Z_2)\},$$

where Z_1 and Z_2 are unions of irreducible components of Z such that $Z = Z_1 \cup Z_2; Z_1, Z_2 \ne \emptyset$. Now, let $\mathscr{F} \ne 0$ be a coherent sheaf over X. In view of (3.2) it is natural to define the *connectedness-subdimension* of $\mathcal{F}$ by the formula

(3.3) $$\underline{c}(\mathcal{F}) := \min\{\underline{\dim}(\bar{T}_1 \cap \bar{T}_2) | T_1 \cup T_2 = \mathrm{Ass}(\mathcal{F}), T_1, T_2 \neq \emptyset\}.$$

To be complete we define $\underline{c}(0) = -1$. We say that a coherent sheaf $\mathcal{F}$ is connected if $\underline{c}(\mathcal{F}) \geq 0$. Obviously we have

(3.4) $$\begin{aligned} &\text{(i)}\ \underline{c}(\mathcal{F}) = \min\{\underline{c}(\mathrm{Supp}(\mathcal{F})), \dim \overline{\{x\}} | x \in \mathrm{Ass}(\mathcal{F})\}, \\ &\text{(ii)}\ \mathcal{F} \text{ connected} \iff \mathrm{Supp}(\mathcal{F}) \text{ connected.} \end{aligned}$$

To relate depths and connectivity for sheaves we prove

(3.5) **Lemma.** *Let A be a noetherian ring, let $I \subseteq A$ be an ideal, and let $M \neq 0$ be an indecomposable finitely generated A-module with $g(I, M) > 1$. Let $T_1, T_2 \subseteq \mathrm{Ass}(M)$ such that $T_1, T_2 \neq \emptyset$, $T_1 \cup T_2 = \mathrm{Ass}(M)$. Then $\bar{T}_1 \cap \bar{T}_2 \not\subseteq V(I)$.*

Proof. Suppose $\bar{T}_1 \cap \bar{T}_2 \subseteq V(I)$. Put $J_i = \bigcap_{\mathfrak{p} \in T_i} \mathfrak{p}$ $(i = 1, 2)$. Then $I \subseteq \sqrt{J_1 + J_2}$. In particular we have $g(J_1 + J_2, M) \geq g(I, M) > 1$, $g(J_i, M) = 0$ $(i = 1, 2)$ and $J_1 \cap J_2 \subseteq \sqrt{\mathrm{ann}(M)}$. Considering the following piece of the Mayer-Vietoris sequence for local cohomology:

$$H^0_{J_1+J_2}(M) \to H^0_{J_1}(M) \oplus H^0_{J_2}(M) \to H^0_{J_1 \cap J_2}(M) \to H^1_{J_1+J_2}(M),$$

we get $H^0_{J_1}(M) \oplus H^0_{J_2}(M) \cong H^0_{J_1 \cap J_2}(M) = M$, $H^0_{J_i}(M) \neq 0$. This contradicts the assumption that M is indecomposable.

We also shall need the following graded version of (3.5), which is shown in the same way using "graded" local cohomology.

(3.5)$'$ **Lemma.** *Let A be a noetherian graded ring, let $I \subseteq A$ be a graded ideal, and let $M \neq 0$ be a finitely generated graded A-module, which is indecomposable in the category of graded A-modules. Let $g(I, M) > 1$ and let $T_1, T_2 \subseteq \mathrm{Ass}(M)$ such that $T_1, T_2 \neq \emptyset$, $T_1 \cup T_2 = \mathrm{Ass}(M)$. Then $\bar{T}_1 \cap \bar{T}_2 \not\subseteq V(I)$.*

As an application of (3.5) we get

(3.6) **Lemma.** *Let X be a noetherian scheme and let $\mathcal{F}$ be a connected coherent sheaf over X such that the stalk $\mathcal{F}_y$ is an indecomposable $\mathcal{O}_{X,y}$-module for each closed point y of* $\mathrm{Supp}(\mathcal{F})$. *Let $Z \subseteq X$ be a closed set, which contains all closed points of* $\mathrm{Supp}(\mathcal{F})$. *Finally assume that $d_Z := \min\{\mathrm{depth}(\mathcal{F}_x) | x \in Z\} > 1$.*

Then the restriction $\mathcal{F}|X - Z$ of $\mathcal{F}$ to the open subset $X - Z$ is connected and satisfies the inequality $\underline{c}(\mathcal{F}|X - Z) \geq d_Z - 2$.

Proof. As $d_Z > 0$ we have $\mathrm{Ass}(\mathcal{F}|X-Z) = \mathrm{Ass}(\mathcal{F})$. Now let $T_1, T_2 \subseteq \mathrm{Ass}(\mathcal{F})$ be nonempty and such that $T_1 \cup T_2 = \mathrm{Ass}(\mathcal{F})$. Denote by $\bar{T}_1$ and $\bar{T}_2$ their closures in X. As $\mathcal{F}$ is connected we have $\bar{T}_1 \cap \bar{T}_2 \neq \emptyset$. So we find a closed point $y \in \bar{T}_1 \cap \bar{T}_2$. By our assumption we have $y \in Z$, and $\mathcal{F}_y$ is indecomposable. Let $I \subseteq \mathcal{O}_{X,y}$ be the vanishing ideal of $\bar{T}_1 \cap \bar{T}_2$ at y. By (3.5) we have $g(I, \mathcal{F}_y) \leq 1$. Let $J \subseteq \mathcal{O}_{x,y}$ be the vanishing ideal of Z at y and let

$\mathfrak{p}$ be a minimal prime divisor of $I + J$. Then clearly we have $g(\mathfrak{p}, \mathscr{F}_y) \geq d_Z$. This induces $\operatorname{ht}(\mathfrak{p}/I) \geq d_Z - 1$; thus $\dim_x(\bar{T}_1 \cap \bar{T}_2) \geq d_Z - 2$ for all closed points x of $\bar{T}_1 \cap \bar{T}_2 - Z$, which satisfy $y \in \overline{\{x\}}$. Making y run through all closed points of $\bar{T}_1 \cap \bar{T}_2$, we obtain $\underline{\dim}(\bar{T}_1 \cap \bar{T}_2 - Z) \geq d_Z - 2$. This proves our claim.

We now return to the morphism $\pi\colon \operatorname{Proj}(S) \to \operatorname{Spec}(R)$ defined in (2.1) and look at the connectivity of the sheaf $\mathscr{F}$ induced by a noetherian, graded, essential S-module $M = \bigoplus_{n \in \mathbb{Z}} M_n$ (cf. (2.2)). Defining $\operatorname{depth}^*(M)$ according to (2.3) we have

(3.7) **Proposition.** *Assume that* $\mathscr{F}$ *is connected and that the stalks of* $\mathscr{F}$ *are indecomposable in all closed points of* $\operatorname{Supp}(\mathscr{F})$. *Let* $\operatorname{depth}^*(M) > 1$. *Then the restriction* $\mathscr{F}|(\operatorname{Proj}(S) - \pi^{-1}(\mathfrak{m}))$ *is connected and satisfies the inequality*

$$\underline{c}(\mathscr{F}|(\operatorname{Proj}(S) - \pi^{-1}(\mathfrak{m}))) \geq \operatorname{depth}^*(M) - 2.$$

Proof. Apply (3.6) with $Z = \pi^{-1}(\mathfrak{m})$ and use (2.5)(ii).

We notice the following criterion for the connectedness of $\mathscr{F}$, in which $S_{\geq 1}$ denotes the irrelevant ideal $S_1 \oplus S_2 \oplus \cdots$ of S.

(3.8) **Proposition.** *Let* M *be indecomposable as a graded module and assume that* $g(S_{\geq 1}, M) > 1$. *Then* $\mathscr{F}$ *is connected.*

Proof. Let $T_1, T_2 \subseteq \operatorname{Ass}(\mathscr{F})$ nonempty and such that $T_1 \cup T_2 = \operatorname{Ass}(\mathscr{F})$. We must show $\bar{T}_1 \cap \bar{T}_2 \neq \varnothing$ (in $\operatorname{Proj}(S)$). Considering T_1 and T_2 as sets of prime ideals this comes up to prove that there are homogeneous prime ideals $\mathfrak{q} \in \operatorname{Spec}(S) - V(S_{\geq 1})$, $\mathfrak{p}_1 \in T_1$, $\mathfrak{p}_2 \in T_2$ with $\mathfrak{p}_1, \mathfrak{p}_2 \subseteq \mathfrak{q}$. This is clear by $(3.5)'$.

(3.9) *Remark.* The condition $g(S_{\geq 1}, M) > 1$ exactly means that there is a canonical isomorphism (cf. [10])

$$M \xrightarrow{\cong} \bigoplus_n H^0(\operatorname{Proj}(S), \mathscr{F}(n)).$$

This observation also immediately gives a proof of (3.8). In the case $M = S$ the connectivity of $\mathscr{F} = \mathscr{O}_{\operatorname{Proj}(S)}$ exactly corresponds to the connectivity of $\operatorname{Proj}(S)$ (see (3.4)(ii)) and thus is a classical subject of algebraic geometry. One of the mostly used criteria for the connectedness of $\operatorname{Proj}(S)$ is the fact that $\operatorname{Proj}(S)$ is connected iff its ring of global sections $H^0(\operatorname{Proj}(S), \mathscr{O}_{\operatorname{Proj}(S)}) = \Gamma$ is a local ring. This is due to the observation that $\operatorname{Spec}(\Gamma) \xrightarrow{\nu} \operatorname{Spec}(R)$ is a Grothendieck-Stein factor of the morphism $\pi\colon \operatorname{Proj}(S) \to \operatorname{Spec}(R)$ [8].

4. Blowing up

In this section let $(R, \mathfrak{m})$ be a local noetherian ring and let $I \subseteq \mathfrak{m}$ be an ideal of positive height. We want to study the blowing up morphism (cf. (1.3))

$$\operatorname{Bl}_R(I) = \operatorname{Proj}(\mathfrak{R}(I)) \xrightarrow{\pi_I} \operatorname{Spec}(R).$$

Thereby we are mainly interested in connectivity of the exceptional fiber

$$F_R(I) := \pi_I^{-1}(V(I)) = \mathrm{Proj}(\mathrm{Gr}(I)) \tag{4.1}$$

of π_I and of the complement

$$C_R(I) := F_R(I) - \pi_I^{-1}(\mathfrak{m}) \tag{4.2}$$

of the special fiber $\pi_I^{-1}(\mathfrak{m})$ in exceptional fiber. First we give the following connectedness-criterion for the exceptional fiber $F_R(I)$, in which $\hat{R}$ denotes the $\mathfrak{m}$-adic completion of R.

(4.3) **Proposition.** *If* $\mathrm{Spec}(\hat{R}) - V(I\hat{R})$ *is connected, the exceptional fiber* $F_R(I)$ *is connected.*

Proof. Assume that $\mathrm{Spec}(\hat{R}) - V(I\hat{R})$ is connected. Let Z_1, Z_2 be unions of irreducible components of $\mathrm{Bl}_R(I\hat{R})$ such that $Z_1 \cup Z_2 = \mathrm{Bl}_{\hat{R}}(I\hat{R})$. As $\mathrm{ht}(I\hat{R}) > 0$ the canonical images $\pi_{I\hat{R}}(Z_1)$, $\pi_{I\hat{R}}(Z_2) \subseteq \mathrm{Spec}(\hat{R})$ are again unions of irreducible components and cover the whole set $\mathrm{Spec}(\hat{R})$. By our connectedness assumption $\pi_{I\hat{R}}(Z_2) \cap \pi_{I\hat{R}}(Z_1)$ contains a point $y \in \mathrm{Spec}(\hat{R}) - V(I\hat{R})$. In particular, y has one single preimage point $x \in \mathrm{Bl}_{\hat{R}}(I\hat{R})$. So x must belong to $Z_1 \cap Z_2$. This shows that $\mathrm{Bl}_{\hat{R}}(I\hat{R})$ is connected. Therefore $F_R(I)$ must be connected (cf. [6, (3.4) or 5, (2.5)]).

(4.4) **Corollary.** *If* $g(I, R) > 1$, *the exceptional fiber* $F_R(I)$ *is connected.*

Proof. Passing to completion we get $g(I\hat{R}, \hat{R}) > 1$. Applying (3.5) with $A = M = \hat{R}$ we see that $\mathrm{Spec}(\hat{R}) - V(I\hat{R})$ is connected and thus may conclude by (4.3).

Now we formulate a connectedness-criterion for the complement $C_R(I)$. To do so, we introduce the set $T(I)$ of all images of generic points of the exceptional fiber:

$$T(I) := \{\pi_I(x) | x = \text{ generic point of } F_R(I)\}. \tag{4.5}$$

Using this notation we have

(4.6) **Proposition.** *Assume that* $g(I, R) > 1$ *or that* R *is an excellent normal domain. Then* $C_R(I)$ *is connected if and only if* $T(I)$ *satisfies the condition*

$(*)$ $\bar{U}_1 \cap \bar{U}_2 - \{\mathfrak{m}\} \neq \emptyset \quad \forall U_1, U_2 \subseteq T(I)$ *with* $U_1, U_2 \neq \emptyset$ *and* $U_1 \cup U_2 = T(I)$.

Proof. Clearly the connectedness of $C_R(I)$ induces the condition $(*)$. To prove the converse, let $T_1 \cup T_2$ be a decomposition of the set of all generic points of $F_R(I)$ such that $\bar{T}_i - \pi_I^{-1}(\mathfrak{m}) \neq \emptyset$ $(i = 1, 2)$. It suffices to show that $\bar{T}_1 \cap \bar{T}_2 - \pi_I^{-1}(\mathfrak{m})$ is not empty. As $T(I)$ satisfies the condition $(*)$, we find a $\mathfrak{p} \in \pi_I(\overline{T}_1) \cap \pi_I(\overline{T}_2) - \{\mathfrak{m}\}$. We claim that the exceptional fiber $F_{R_\mathfrak{p}}(I_\mathfrak{p})$ of the localized blowing-up $\mathrm{Bl}_{R_\mathfrak{p}}(I_\mathfrak{p}) \to \mathrm{Spec}(R_\mathfrak{p})$ is connected. This follows from (4.4), respectively from (4.3), as either $g(I_\mathfrak{p}, R_\mathfrak{p}) > 1$ or as $(R_\mathfrak{p})^\wedge$ is a domain (cf. [9]).

If we consider $F_{R_\mathfrak{p}}(I_\mathfrak{p})$ as a subset of $F_R(I)$, the observed connectedness furnishes a point $x \in F_{R_\mathfrak{p}}(I_\mathfrak{p}) \cap \bar{T}_1 \cap \bar{T}_2$. As $F_{R_\mathfrak{p}}(I_\mathfrak{p}) \cap \pi_I^{-1}(\mathfrak{m}) \neq \varnothing$, we get our claim.

Now, we want to give an estimate on the *small connectedness-subdimension* $\dot{\underline{c}}(C_R(I))$ of $C_R(I)$, which we define by

(4.7) $$\dot{\underline{c}}(C_R(I)) := \underline{c}(\mathscr{O}_{\mathrm{Proj}(\mathrm{Gr}(I))}|C_R(I)) \leq \underline{c}(C_R(I)).$$

Defining $\bar{t}(I)$ according to (1.1)(ii) we get

(4.8) **Proposition.** *Let* $\bar{t}(I) > 1$ *and assume that either* $\mathrm{Spec}(\hat{R}) - V(I\hat{R})$ *is connected or that* $g(I, R) > 1$. *Then* $C_R(I)$ *is connected and satisfies*

$$\dot{\underline{c}}(C_R(I)) \geq \bar{t}(I) - 2.$$

Proof. By (4.3) or by (4.4), $F_R(I)$ is connected. Now we apply (3.7) with $S = M = \mathrm{Gr}(I)$.

(4.9) **Example.** Let $(R, \mathfrak{m})$ be a local Cohen–Macaulay ring and let $\mathfrak{p} \in \mathrm{Spec}(R)$ be an almost complete intersection of height $h > 1$, which is a generic complete intersection. Thus, denoting by $\mu_\mathfrak{q}(\mathfrak{p})$ the minimal number of generators of the localized ideal $\mathfrak{p}_\mathfrak{q} \subseteq R_\mathfrak{q}$ ($\mathfrak{q} \in \mathrm{Spec}(R)$), we have $\mu_\mathfrak{m}(\mathfrak{p}) = h+1$, $\mu_\mathfrak{p}(\mathfrak{p}) = h$. We put

$$U(\mathfrak{p}) := \{\mathfrak{q} \in V(\mathfrak{p}) \mid \mathrm{ht}(\mathfrak{q}/\mathfrak{p}) = 1,\ \mu_\mathfrak{q}(\mathfrak{p}) = h + 1\}.$$

By [4] we know

(i) $\bar{t}(\mathfrak{p}) = \min\{\dim(R/\mathfrak{p}) - 1, \mathrm{depth}(R/\mathfrak{p})\}$,

(ii) *if* $\mathfrak{q} \in V(\mathfrak{p})$, $\tilde{\mathfrak{q}} := \mathfrak{q}\ \mathrm{Gr}(I)_\mathfrak{q} \cap \mathrm{Gr}(I)$ *is the unique minimal prime divisor of* $\mathfrak{q}\ \mathrm{Gr}(I)$ *which retracts to* $\mathfrak{q}$.

(iii) $\mathfrak{q} \to \tilde{\mathfrak{q}}$ *defines a* 1-1 *correspondence*

$$U(\mathfrak{p}) \cup \{\mathfrak{p}\} \xleftrightarrow{\sim} \{x \in F_R(\mathfrak{p}) | x \text{ generic }\} = \mathrm{Ass}(\mathscr{O}_{\mathrm{Proj}(\mathrm{Gr}(\mathfrak{p}))}).$$

By (4.6) and (4.8) we get from (i) and (iii)

(iv) (a) $C_R(\mathfrak{p})$ *is connected.*

(b) $\dot{\underline{c}}(C_R(\mathfrak{p})) = \underline{c}(C_R(\mathfrak{p})) \geq \min\{\dim(R/\mathfrak{p}) - 3, \mathrm{depth}(R/\mathfrak{p}) - 2\}$.

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