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SYZYGY PAIRS IN A MONOMIAL ALGEBRA

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ABSTRACT. In this paper we construct the set of "syzygy pairs" of a finite-dimensional monomial algebra and use it to prove the following theorem.

Theorem A. (Finitistic global dimension theorem) *Let Λ be a monomial (zero relations) algebra over a field k. Let M be a Λ-module with finite injective dimension. Then*

$$\text{inj dim}_\Lambda M \leq \dim_k \text{rad } \Lambda$$

This is an improvement on [GKK] who were the first to show that monomial algebras satisfy the finitistic global dimension conjecture.

Conjecture. (A. Rosenberg and D. Zelinsky) Let Λ be a finite-dimensional algebra over a field k. Then the injective dimension of the Λ-modules with finite injective dimension is bounded.

One immediate consequence of Theorem A is the following:

Corollary B. *Let Λ be a monomial algebra with finite global dimension. Then*

$$\text{gldim } \Lambda \leq \dim_k \text{rad } \Lambda. \quad \square$$

The following example shows that, in some sense this is the best possible bound.

Example. Let Λ be the quotient of the $(n+1)\times(n+1)$ lower triangular matrix ring with entries in k by the square of its radical. In other words Λ is given by the quiver A_{n+1}:

$$\bullet_{v_0} \to \bullet_{v_1} \to \cdots \to \bullet_{v_n}$$

with the relations: any path of length ≥ 2 is zero. Then gldim $\Lambda = n = \dim_k \text{rad } \Lambda$.

We briefly recall the definition of a finite-dimensional monomial algebra. Let Q be a finite directed graph. Then kQ, the path algebra of Q, is the algebra spanned as a vector space by all the directed paths in Q. The multiplication of two paths is their composition, or zero if they are not composable. A monomial

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algebra is a quotient of kQ by an ideal generated by paths of length at least two. We refer the reader to [GHZ] for more details.

Throughout this paper, Λ will denote a finite-dimensional monomial algebra. The modules are left Λ modules.

THE MAIN THEOREM

The injective dimension of M is the largest integer n such that $\mathrm{Ext}^n_\Lambda(S, M) \neq 0$ for some simple Λ-module S. The problem is now very easy since the minimal projective resolution of a simple module has a very nice form. There are only a finite number of indecomposable modules which occur as summands of syzygies of simple modules. (We call these modules K_γ.) Therefore after a finite number of steps, the projective resolution of S starts repeating and this leads to a bound on the dimension of $\mathrm{Ext}^*_\Lambda(S, M)$.

We now construct the modules K_γ which occur as summands of syzygies of simple modules. If Q is the quiver of Λ, let S_v denote the simple module at the vertex v, and let P_v be its projective cover.

Let γ be a path in the quiver of Λ of length ≥ 1 so that γ contains no zero relations (i.e. $\gamma \neq 0$ in Λ). Suppose that γ starts at v and ends at w. Then γ gives a homomorphism of Λ modules $\gamma^*: P_w \to P_v$ which is nonzero at w. Let $K_\gamma \subseteq P_v$ be the image of this homomorphism. Then the pair (P_v, K_γ) will be called a *syzygy pair* for Λ. A homomorphism of syzygy pairs is defined in the obvious way.

Lemma 1.1. (a) *Syzygy pairs corresponding to different paths are not isomorphic.*

(b) *The number of isomorphism classes of syzygy pairs equals the dimension of* rad Λ *as a* k*-vector space.*

Proof. (b) follows from (a) because the nonzero paths of length ≥ 1 form a basis of the radical of Λ. To prove (a) suppose that $(P_u, K_\alpha) \simeq (P_v, K_\beta)$. Then $u = v$ and α, β must be paths from v to the same vertex, say w. The isomorphism $K_\alpha \simeq K_\beta$ lifts to an isomorphism $P_w \simeq P_w$ making the following diagram commute:

$$\begin{array}{ccc} P_w & \xrightarrow{\alpha^*} & P_v \\ f\big\downarrow\simeq & & \simeq\big\downarrow g \\ P_w & \xrightarrow{\beta^*} & P_v \end{array}$$

But then

$$\begin{aligned} g\alpha^*(e_w) &= a\alpha + \sum a'(\text{longer paths}) \;= \beta^* f(e_w) \\ &= b\beta + \sum b'(\text{longer paths}) \text{ where } a,\ b \in k^* \text{ and } a',\ b' \in k. \end{aligned}$$

This implies that $\alpha = \beta$. $\square$

The following was essentially proved in [GHZ].

Lemma 1.2. *Let $\gamma\colon v \to w$ be a path of length ≥ 1 in the quiver of Λ. Then the kernel of the induced map $\gamma^*\colon P_w \to P_v$ is the direct sum of the submodules K_{γ_i} of P_w where γ_i ranges over all paths satisfying the following conditions*:

(1) *$\gamma_i\colon w \to u_i$ is a path starting at w.*

(2) *$\gamma\gamma_i\colon v \to u_i$ contains exactly one (minimal) zero relation ending at u_i.*

Proof. Let $\widetilde{Q}$ be the universal covering of the quiver Q of Λ. Then $\widetilde{Q}$ is an infinite tree (or disjoint union of trees) and all the zero relations in Q lift to $\widetilde{Q}$. Let $\tilde{\gamma}\colon \tilde{v} \to \tilde{w}$ be a lifting of γ. Then we have an induced map $\tilde{\gamma}^*\colon P_{\tilde{w}} \to P_{\tilde{v}}$ and the kernel of γ^* is the push down of the kernel of $\tilde{\gamma}^*$. Thus we are reduced to finding the summands of $\operatorname{Ker} \tilde{\gamma}^* \subseteq P_{\tilde{w}}$. The result follows if we use the following well-known fact concerning representations of trees with relations: if X is a representation with simple top, then Y is a direct sum of representations with simple top, where $0 \to Y \to P \to X \to 0$ is exact and P is the projective cover representation of X. It is now clear that $\operatorname{Ker} \gamma^* = \amalg K_{\gamma_i}$, where γ_i satisfy conditions (1) and (2). □

Definition 1.3. If (P_v, K_γ) is a syzygy pair, let $\Omega^1(P_v, K_\gamma)$ denote the set of all syzygy pairs (P_w, K_{γ_i}) where the γ_i are as given in Lemma 1.2. Then, for each pair (P_w, K_{γ_i}) of $\Omega^1(P_v, K_\gamma)$ we construct $\Omega^1(P_w, K_{\gamma_i})$ by applying 1.2 and we denote by $\Omega^2(P_v, K_\gamma)$ the union of all $\Omega^1(P_w, K_{\gamma_i})$. Inductively, let $\Omega^n(P_v, K_\gamma)$ be the union of the sets $\Omega^{n-1}(P_w, K_{\gamma_i})$ for $n \geq 2$.

Lemma 1.4. *Let $(P, K) \in \Omega^n(P', K')$ and let $\cdots \to P_n(K') \xrightarrow{d_n} P_{n-1}(K') \to \cdots \to P_0(K') \to K' \to 0$ be a minimal projective resolution of K'. Then there is a summand A of $P_{n-1}(K')$ and a summand B of $d_n P_n(K')$ such that $B \subseteq A$ and $(A, B) \simeq (P, K)$.*

Proof. This follows from Lemma 1.2 by induction on n. □

Definition 1.5. We say that a syzygy pair (P, K) is *periodic* if (P, K) is isomorphic to an element of $\Omega^n(P, K)$ for some $n \geq 1$. The smallest such n is called the *period* of (P, K). We say that (P, K) is *virtually periodic* if (P, K) is isomorphic to an element of $\Omega^n(P', K')$ for some periodic pair (P', K') and some $n \geq 1$.

Note that every periodic syzygy is virtually periodic.

Lemma 1.6. *Suppose that (P, K) is a virtually periodic syzygy pair and M is a Λ-module with finite injective dimension. Then any homomorphism $K \to M$ extends to a homomorphism on P.*

Proof. Let (P', K') be a periodic syzygy pair of period p so that $(P, K) \in \Omega^n(P', K')$. Then $(P, K) \in \Omega^{n+pm}(P', K')$ for all $m \geq 1$. But $\operatorname{Ext}_\Lambda^{n+pm}(K', M) = 0$ for sufficiently large m, so $\operatorname{Hom}_\Lambda(P, M)$ must map onto $\operatorname{Hom}_\Lambda(K, M)$ by Lemma 1.4. □

Lemma 1.7. *Let* (P_0, K_0) *be any syzygy pair, and let* $n \geq \dim_k \operatorname{rad} \Lambda$. *Then every element of* $\Omega^n(P_0, K_0)$ *is virtually periodic.*

Proof. Let $(P_n, K_n) \in \Omega^n(P_0, K_0)$. From the definition of Ω we see that there is a sequence of pairs $(P_1, K_1), \dots, (P_{n-1}, K_{n-1})$ so that (P_{i+1}, K_{i+1}) belongs to $\Omega(P_i, K_i)$ for all $i = 0, 1, \dots, n-1$. Since there are $\dim_k \operatorname{rad} \Lambda$ distinct isomorphism classes of syzygy pairs (1.1), the same pair must occur twice in the sequence. That pair is periodic, so the last pair (P_n, K_n) must be virtually periodic. □

We are ready to prove our main result.

Proof of Theorem A. Suppose M has finite injective dimension. An immediate application of Lemmas 1.6 and 1.7 shows that $\operatorname{Ext}^n_\Lambda(K, M) = 0$ for any syzygy pair (P, K) and $n \geq \dim_k \operatorname{rad} \Lambda$. But the radical of P_v is the direct sum of all K_γ where γ runs over all the arrows (paths of length 1) starting at v. Thus $\operatorname{Ext}^{n+1}_\Lambda(S_v, M) = 0$ for every simple module S_v and

$$\operatorname{inj} \dim_\Lambda M \leq \dim_k \operatorname{rad} \Lambda. \quad \square$$

Another immediate application of our results is the following:

Proposition 1.8. *A monomial algebra has infinite global dimension if and only if there is a periodic syzygy pair.*

References

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