STABLE UNIT BALLS IN ORLICZ SPACES

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(Communicated by William J. Davis)

ABSTRACT. Let $L^\phi(\mu)$ be an Orlicz space and $X \subseteq L^\phi(\mu)$ an ideal such that $I_\phi(f/\|f\|) = 1$ for each $f \in X \setminus \{0\}$. Then the unit ball B_X is stable, that is, the midpoint map $\Phi_{1/2}: B_X \times B_X \rightarrow B_X$ defined by $\Phi_{1/2}(x, y) = \frac{1}{2}(x + y)$, is open. In particular, $B_E\phi$ is stable, E^ϕ being the subspace of finite elements of $L^\phi(\mu)$ (i.e., $f \in E^\phi$ iff $I_\phi(\lambda f) < +\infty$ for each $\lambda > 0$), and $B_{L^\phi(\mu)}$ is stable when ϕ satisfies condition (Δ_2) or (δ_2) , depending on the measure μ .

If X is a Banach space, we put B_X and S_X for the closed unit ball and unit sphere of X , respectively. If $x \in X$ and $\varepsilon > 0$, then $B(x; \varepsilon) = \{y \in X: \|x - y\| \leq \varepsilon\}$. For a set C , denote by $\text{Ext. } C$, the set of its extreme points. A convex subset C of a topological vector space E is said *stable* if the midpoint map $\Phi_{1/2}: C \times C \rightarrow C$ such that $\Phi_{1/2}(x, y) = \frac{1}{2}(x + y)$, is open. Compact stable sets have been studied in [2, 8]. Many Banach spaces have stable unit ball, namely, Banach spaces with 3.2.I.P. [2, p. 195] and finite dimensional Musielak-Orlicz spaces [3]. A Banach space X with stable unit ball has interesting properties, for example:

- (i) The set $\text{Ext. } B_X$ is closed.
- (ii) If K is a compact set and $C(K, X)$ denotes the Banach space of continuous functions $f: K \rightarrow X$, then $f \in \text{Ext. } B_{C(K, X)}$ iff $f(k) \in \text{Ext. } B_X$, for each $k \in K$.

In [4] Grzaslewicz proved that Orlicz spaces $L^\phi(\mu)$, when ϕ fulfills good conditions (as (Δ_2) or (δ_2)), satisfy (ii). Here this result and others follow from the fact that the unit ball $B_{L^\phi(\mu)}$ of the Orlicz space $L^\phi(\mu)$ is stable when ϕ is (Δ_2) or (δ_2) , depending on μ .

We consider a positive measure space (Ω, Σ, μ) and an Orlicz function $\phi: \mathbf{R} \rightarrow [0, \infty)$ convex, even, $\phi(0) = 0$ and $\phi(x) > 0$ for some $x \in \mathbf{R}$. The Orlicz space $L^\phi(\mu)$ is the set of (equivalence classes of) measurable functions

Received by the editors November 17, 1988.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 46E30; Secondary 46B20, 47D20.

Key words and phrases. Stable sets, Orlicz spaces, unit ball.

Supported in part by DGICYT Grant PB88-0141.

$f: \Omega \rightarrow \mathbf{R}$ such that the functional $I_\phi(\lambda f) = \int_\Omega \phi(\lambda f) d\mu < +\infty$, for some $\lambda > 0$, equipped with Luxemburg norm $\|f\| = \inf\{\lambda > 0: I_\phi(f/\lambda) \leq 1\}$.

The function ϕ satisfies condition (Δ_2) if there are two constants $b > 0$, $k > 0$ such that $\phi(2u) \leq k\phi(u)$, $|u| \geq b$. The function ϕ satisfies condition (δ_2) if there are two constants $a > 0$, $k > 0$ such that $\phi(a) > 0$ and $\phi(2u) \leq k\phi(u)$, $|u| \leq a$.

Let U_ϕ be the maximal open subset of $\mathbf{R}$ such that ϕ is affine on each connected component of U_ϕ . The set U_ϕ consists of a countable (finite or infinite) number of connected components. In general, if ω is the first infinite ordinal, there exists $\beta \leq \omega$ such that

$$U_\phi = (-a_0, a_0) \cup \left\{ \bigcup_{n < \beta} \{(-b_n, -a_n) \cup (a_n, b_n)\} \right\},$$

$$0 \leq a_0 \leq a_n < b_n, \quad n \geq 1,$$

where $a_0 = \sup\{u \geq 0: \phi(u) = 0\}$ and $\{(a_n, b_n)\}_{n < \beta}$ is the countable family of intervals of $\mathbf{R}^+$ where ϕ is affine. We put m_n for the gradient of ϕ on (a_n, b_n) , $n \geq 1$.

If X is a Banach lattice, an ideal of X is a closed linear subspace $Y \subseteq X$ such that if $x \in X$, $y \in Y$ and $|x| \leq |y|$, then $x \in Y$. In the following we use the notation $x \vee y = \sup\{x, y\}$, $x \wedge y = \inf\{x, y\}$ and χ_A for the characteristic function of the set A .

1. **Lemma.** Let (Ω, Σ, μ) be a measure space, $L^\phi(\mu)$ an Orlicz space and $Y \subseteq L^\phi(\mu)$ an ideal. Then:

- (I) If $f \in Y$ and $A \in \Sigma$, we have $\chi_A \cdot f \in Y$ and the function $G: Y \rightarrow Y$ such that $G(f) = \chi_A \cdot f$, $f \in Y$, is continuous.
- (II) Let $c, d: \Omega \rightarrow \mathbf{R}$ be measurable functions and $h \in Y$, $0 \leq h$. For each $g \in L^\phi(\mu)$ define $F(g) = h \wedge [0 \vee [(d - g) \wedge (g - c)]]$. Then $F(g) \in Y$ and $F: L^\phi(\mu) \rightarrow Y$ is continuous.

Proof. (I). Since $|\chi_A \cdot f| \leq |f|$, clearly $\chi_A \cdot f \in Y$. To prove continuity, observe that, for $f, g \in Y$, $|\chi_A \cdot f - \chi_A \cdot g| \leq |f - g|$. So $\|\chi_A \cdot f - \chi_A \cdot g\| \leq \|f - g\|$.

(II). By definition of the function F , $F(g)$ is measurable and $0 \leq F(g) \leq h$, for each $g \in L^\phi(\mu)$. Hence $F(g) \in Y$. Put

$$H(g_1, g_2) = h \wedge (d - g_1) \wedge (g_2 - c), \quad g_1, g_2 \in L^\phi(\mu).$$

As real functions, we have (see [5, p. 1])

$$|H(g_1, g_2) - H(g_2, g_2)| \leq |g_1 - g_2| \leq |H(g_1, g_2) - H(g_1, g_1)|.$$

Then $|H(g_1, g_1) - H(g_2, g_2)| \leq 2|g_1 - g_2|$. Therefore, for $g_1, g_2 \in L^\phi(\mu)$,

$$\begin{aligned} |F(g_1) - F(g_2)| &= |0 \vee H(g_1, g_1) - 0 \vee H(g_2, g_2)| \\ &\leq |H(g_1, g_1) - H(g_2, g_2)| \leq 2|g_1 - g_2|. \end{aligned}$$

Thus $\|F(g_1) - F(g_2)\| \leq 2\|g_1 - g_2\|$ which implies continuity of F . $\square$

2. **Lemma.** Let $L^\phi(\mu)$ be an Orlicz space, $A \in \Sigma$ and define (where it exists) $F_A(f) = \int_A f d\mu$, for each $f \in L^\phi(\mu)$. Then:

- (1) If $\mu(A) < +\infty$, $F_A \in (L^\phi(\mu))^*$ (= dual of $L^\phi(\mu)$).
- (2) If $A = \bigcup_{n \geq 1} A_n$, disjoint partition, with $\mu(A_n) < \infty$, $n \geq 1$, and there exists a measurable function $g: \Omega \rightarrow \mathbf{R}$, $g \geq 0$, such that $\int_A g d\mu < +\infty$, then F_A is continuous on the set $\{f \in L^\phi(\mu): -g \leq f \leq g\}$.

Proof. (1). This result is true in Köthe functions spaces. So it is true for Orlicz spaces (see [5, p. 28])

(2). This follows immediately from (1).

3. **Proposition.** Let $L^\phi(\mu)$ be an Orlicz space and $X \subseteq L^\phi(\mu)$ an ideal such that, for each $f \in X \setminus \{0\}$, $I_\phi(f/\|f\|) = 1$. Then B_X is stable.

Proof. Take $f_i \in B_X$, $i = 0, 1, 2$, with $f_0 = \frac{1}{2}(f_1 + f_2)$ and $V(f_i)$ a neighborhood of f_i , $i = 1, 2$, in B_X . We prove that $2^{-1}(V(f_1) + V(f_2))$ is a neighborhood of f_0 in B_X .

Suppose that $\|f_0\| < 1$. In this case, the following discussion is valid for any Banach space. As $\|f_0\| < 1$, we can choose $\varepsilon > 0$ and $g_i \in V(f_i)$, $i = 1, 2$, such that $B(g_i; \varepsilon) \subseteq V(f_i)$ and $2^{-1}(g_1 + g_2) = f_0$. Hence

$$B(f_0; \varepsilon) = 2^{-1}(B(g_1; \varepsilon) + B(g_2; \varepsilon)) \subseteq 2^{-1}(V(f_1) + V(f_2))$$

Therefore $2^{-1}(V(f_1) + V(f_2))$ is a neighborhood of f_0 in B_X .

Assume now that $\|f_0\| = 1$. In this case it is enough to find a continuous function $S = (S_1, S_2): B_X \rightarrow B_X \times B_X$ such that $S_i(f_0) = f_i$, $i = 1, 2$, and $\Phi_{1/2} \circ S = \text{identity on } B_X$, $\Phi_{1/2}$ being the midpoint map $\Phi_{1/2}(u, v) = 2^{-1}(u + v)$. Indeed, $S^{-1}(V(f_1) \times V(f_2))$ will be a neighborhood of f_0 in B_X and $S^{-1}(V(f_1) \times V(f_2)) \subseteq 2^{-1}(V(f_1) + V(f_2))$. For $w \in \Omega$ we have $\phi(f_0(w)) \leq 2^{-1}(\phi(f_1(w)) + \phi(f_2(w)))$ a.e. because ϕ is convex. Since $\|f_0\| = 1$, then $\|f_1\| = 1 = \|f_2\|$. From

$$1 = I_\phi(f_0) \leq 2^{-1}(I_\phi(f_1) + I_\phi(f_2)) = 1,$$

we obtain that $\phi(f_0(w)) = 2^{-1}(\phi(f_1(w)) + \phi(f_2(w)))$ a.e., that is, either $f_0(w) = f_i(w)$ or $f_0(w)$ and $f_i(w)$, $i = 1, 2$, belong to the closure of the same connected component of U_ϕ a.e. Put

$$\begin{aligned} A^+ &= \{w \in \Omega: f_1(w) > f_0(w), f_0(w) \in (-a_0, a_0)\} \\ A^- &= \{w \in \Omega: f_1(w) < f_0(w), f_0(w) \in (-a_0, a_0)\}, \quad A = A^+ \cup A^- \end{aligned}$$

and, for $n \geq 1$,

$$U_n^+ = \{w \in \Omega: f_1(w) > f_0(w), f_0(w) \in (a_n, b_n)\}, \quad U^+ = \bigcup_{n \geq 1} U_n^+$$

$$U_n^- = \{w \in \Omega: f_1(w) < f_0(w), f_0(w) \in (-b_n, -a_n)\}, \\ U^- = \bigcup_{n \geq 1} U_n^-, \quad U = U^+ \cup U^-$$

$$L_n^+ = \{w \in \Omega: f_1(w) < f_0(w), f_0(w) \in (a_n, b_n)\}, \quad L^+ = \bigcup_{n \geq 1} L_n^+$$

$$L_n^- = \{w \in \Omega: f_1(w) > f_0(w), f_0(w) \in (-b_n, -a_n)\}, \\ L^- = \bigcup_{n \geq 1} L_n^-, \quad L = L^+ \cup L^-$$

$$S_n = U_n^+ \cup L_n^+,$$

$$T_n = L_n^- \cup U_n^-$$

Define functions $d, c: \Omega \rightarrow \mathbf{R}$:

$$d = a_0 \cdot \chi_A + \sum_{n \geq 1} b_n \cdot \chi_{S_n} + \sum_{n \geq 1} (-a_n) \cdot \chi_{T_n} \\ c = -a_0 \cdot \chi_A + \sum_{n \geq 1} a_n \cdot \chi_{S_n} + \sum_{n \geq 1} (-b_n) \cdot \chi_{T_n}$$

and, for each $g \in X$, put $F(g) = |f_1 - f_0| \wedge [0 \vee [(d - g) \wedge [g - c]]]$. Observe that $F(f_0) = f_1 - f_0$ on $U^+ \cup A^+ \cup L^-$, $F(f_0) = f_0 - f_1$ on $U^- \cup L^+ \cup A^-$ and $F(f_0) = 0$ outside of these sets. By 1. Lemma, $F: X \rightarrow X$ is continuous just as functions R_i , $i = 1, 2$, defined by

$$g \in X, \quad R_i(g) = g - (-1)^i \cdot [\chi_{U^+ \cup A^+ \cup L^-} - \chi_{U^- \cup A^- \cup L^+}] \cdot F(g).$$

Also we have $R_i(f_0) = f_i$, $i = 1, 2$, and $g = 2^{-1}(R_1(g) + R_2(g))$, for each $g \in X$. To get that $R_i(g) \in B_X$ if $g \in B_X$, we must modify R_i . Define $P, Q: B_X \rightarrow \mathbf{R}$ by

$$g \in B_X, \quad P(g) = \int_U [\phi(R_1(g)) - \phi(g)] d\mu, \\ Q(g) = \int_L [\phi(g) - \phi(R_1(g))] d\mu.$$

We claim that P, Q are continuous on B_X . We study function P ; function Q is similar. For $g \in B_X$ and $n \geq 1$, write $U_n^+(g) = \chi_{U_n^+} \cdot F(g)$ and $U_n^-(g) = \chi_{U_n^-} \cdot F(g)$. Note that, on $U_n^+ \cup U_n^-$,

$$0 \leq \phi[R_1(g)] - \phi(g) = \phi[g + U_n^+(g) - U_n^-(g)] - \phi(g) = m_n \cdot F(g) \\ \leq m_n \cdot |f_1 - f_0| = \phi(f_1) - \phi(f_0),$$

where m_n is the gradient of ϕ in (a_n, b_n) . Also $0 \leq \phi(f_1) - \phi(f_0)$ on U and $\int_U [\phi(f_1) - \phi(f_0)] d\mu < \infty$. So continuity of P follows from the continuity of the function

$$g \rightarrow \int_{U_n^+ \cup U_n^-} F(g) d\mu, \quad g \in B_X,$$

and to prove this continuity it is enough, by 2. Lemma, to prove that $U_n^+ \cup U_n^-$ is σ -finite. Suppose that $\mu(U_n^-) = +\infty$. Then necessarily $\phi(a_n) = 0$, since, if $\phi(a_n) > 0$, we would obtain

$$\int_{\Omega} \phi(f_0) d\mu \geq \int_{U_n^+} \phi(f_0) d\mu \geq \int_{U_n^+} \phi(a_n) d\mu = +\infty.$$

Take a decreasing real sequence $\{t_k\}_{k \geq 1}$ such that $a_n < t_k < b_n$, $t_k \downarrow a_n$, and put $H_k = \{w \in U_n^+ : f_0(w) \in [t_k, b_n)\}$, $k = 1, 2, \dots$. Then $U_n^+ = \bigcup_{k \geq 1} H_k$ and $\mu(H_k) < +\infty$. Analogously, U_n^- is also σ -finite.

Since $R_i(f_0) = f_i$, $i = 1, 2$, we have that $P(f_0) = Q(f_0)$. If $P(f_0) = 0$, then $\mu(U) = 0 = \mu(L)$ and, for each $g \in B_X$, $P(g) = 0 = Q(g)$. Hence $I_{\phi}(R_i(g)) = I_{\phi}(g) \leq 1$ and $R_i(g) \in B_X$, $i = 1, 2$. So function $S = (R_1, R_2)$ holds.

Suppose that $P(f_0) > 0$ and let $V(f_0)$ be the neighborhood

$$V(f_0) = \{g \in B_X : P(g) > 2^{-1} P(f_0) < Q(g)\}$$

and $\psi: B_X \rightarrow [0, 1]$ a continuous function such that $\psi(g) = 0$, if $g \in B_X \setminus V(f_0)$, and $\psi(f_0) = 1$. Define $S_i(g)$, $g \in B_X$, $i = 1, 2$, by

$$S_i(g) = g - (-1)^i \psi(g) \left[\min \left[\frac{Q(g)}{P(g)}, 1 \right] (\chi_{U^+ \cup A^+} - \chi_{U^-}) - \min \left[\frac{P(g)}{Q(g)}, 1 \right] (\chi_{L^+ \cup A^-} - \chi_{L^-}) \right] \cdot F(g)$$

Observe that:

- (1) $S = (S_1, S_2)$ is continuous, $S_i(f_0) = f_i$, $i = 1, 2$, and, for each $g \in B_X$, $2^{-1}[S_1(g) + S_2(g)] = g$.
- (2) If $g \in B_X$, then $I_{\phi}(g) = I_{\phi}(S_i(g))$, $i = 1, 2$, so that $S_i(g) \in B_X$. Indeed, if $g \notin V(f_0)$, it is clear because $S_i(g) = g$ in this case. Suppose that $g \in V(f_0)$ and, for example, that $P(g) \geq Q(g)$. Then

$$\begin{aligned} \int_{\Omega} \phi(S_1(g)) d\mu &= \int_{\Omega} \phi(g) d\mu \\ &+ \int_U \left\{ \phi \left[g + \psi(g) \cdot \frac{Q(g)}{P(g)} (\chi_{U^+} - \chi_{U^-}) F(g) \right] - \phi(g) \right\} d\mu \\ &+ \int_L \left\{ \phi [g - \psi(g) (\chi_{L^+} - \chi_{L^-}) F(g)] - \phi(g) \right\} d\mu \\ &= \int_{\Omega} \phi(g) d\mu + \psi(g) \cdot \frac{Q(g)}{P(g)} \cdot P(g) - \psi(g) \cdot Q(g) = \int_{\Omega} \phi(g) d\mu \end{aligned}$$

Analogously, $I_{\phi}(S_2(g)) = I_{\phi}(g)$. $\square$

4. *Remark.* Note that in 3. Proposition it is proved that if $L^\phi(\mu)$ is an Orlicz space and $f_i \in B_{L^\phi(\mu)}$, $i = 0, 1, 2$, $f_0 = 2^{-1}(f_1 + f_2)$, and $V(f_1), V(f_2)$ are neighborhoods of f_1, f_2 , respectively, in $B_{L^\phi(\mu)}$, then $2^{-1}(V(f_1) + V(f_2))$ is a neighborhood of f_0 when $\|f_0\| < 1$ or $I_\phi(f_0) = 1$.

5. **Corollary.** Let $L^\phi(\mu)$ be an Orlicz space.

- (1) If E^ϕ is the subspace of finite elements of $L^\phi(\mu)$, that is, $E^\phi = \{f \in L^\phi(\mu) : I_\phi(\lambda f) < \infty, \text{ for each } \lambda > 0\}$, then B_{E^ϕ} is stable.
- (2) If ϕ verifies condition (Δ_2) or (δ_2) , depending on μ (see [7], 8.14. Theorem, p. 53), so that $E^\phi = L^\phi(\mu)$, then $B_{L^\phi(\mu)}$ is stable.

Proof. The subspace E^ϕ is an ideal of $L^\phi(\mu)$ such that $I_\phi(f/\|f\|) = 1$ for each $f \in E^\phi \setminus \{0\}$. Now apply 3. Proposition. $\square$

6. **Corollary.** Let $L^\phi(\mu)$ be an Orlicz space and $X \subseteq L^\phi(\mu)$ an ideal such that $I_\phi(f/\|f\|) = 1$, for each $f \in X \setminus \{0\}$. Then:

- (1) If K is a compact Hausdorff and $f \in C(K, X)$, the following are equivalent: (a) $f \in \text{Ext. } B_{C(K, X)}$; (b) $f(k) \in \text{Ext. } B_X$ for each $k \in K$.
- (2) If K is a Choquet simplex (see [1, p. 84]) and $f \in A(K, X)$ (= Banach space of affine continuous functions $g: K \rightarrow X$), the following are equivalent: (a) $f \in \text{Ext. } B_{A(K, X)}$; (b) $f(k) \in \text{Ext. } B_X$, for each $k \in \text{Ext. } K$.

Proof. (1) $(a \Rightarrow b)$. Since B_X is stable, the map $x \rightarrow \{z \in B_X : \text{there exists } y \in B_X \text{ with } 2^{-1}(z + y) = x\}$ is lower semicontinuous on B_X . Hence, applying the Michael's selection theorem [6], if $f(k) \notin \text{Ext. } B_X$ for some $k \in K$, we can represent f as a nontrivial convex combination $f = 2^{-1}(g + h)$, $g, h \in B_{C(K, X)}$.

$(b \Rightarrow a)$. This follows immediately.

(2). This is consequence of 2.1. Theorem [2]. $\square$

7. *Remark.* Suppose that $L^\phi(\mu)$ is an Orlicz space such that $B_{L^\phi(\mu)}$ is stable. Does this condition imply that $E^\phi = L^\phi(\mu)$? If (Ω, Σ, μ) has some infinite atom and $a_0 = \sup\{t \geq 0 : \phi(t) = 0\}$ satisfies $a_0 > 0$, then always $E^\phi \neq L^\phi(\mu)$ but $B_{L^\phi(\mu)}$ might be stable. For instance, if (Ω, Σ, μ) reduces to a family $\{T_i\}_{i \in I}$ of infinite atoms, then $E^\phi = \{0\}$ and $L^\phi(\mu)$ is isometric to $\ell^\infty(I)$, whose closed unit ball $B_{\ell^\infty(I)}$ is stable because $\ell^\infty(I)$ is 3.2.I.P.

But if ϕ is strictly convex, then $E^\phi = L^\phi(\mu)$ and $B_{L^\phi(\mu)}$ stable are equivalent, as is proved below. We need the following well-known Lemma.

8. **Lemma.** Let (Ω, Σ, μ) be a measure space and ϕ a strictly convex Orlicz function. Then for $e \in L^\phi(\mu)$ the following are equivalent:

- (1) $e \in \text{Ext. } B_{L^\phi(\mu)}$;
- (2) $I_\phi(e) = 1$.

Proof. (1 $\Rightarrow$ 2). Suppose $I_\phi(e) \leq 1 - \varepsilon$, $0 < \varepsilon \leq 1$. We can find $0 < t_0 < \infty$ and $A \in \Sigma$, $A \subseteq \{w \in \Omega: |e(w)| \leq t_0\}$ such that $0 < \mu(A) < +\infty$. Choose $\delta > 0$ verifying $\int_A \phi(e \pm \delta) d\mu \leq \int_A \phi(e) d\mu + \varepsilon$ and define $e_1 = e + \delta \cdot \chi_A$, $e_2 = e - \delta \cdot \chi_A$. Then clearly $\|e_i\| \leq 1$, $e = \frac{1}{2}(e_1 + e_2)$ but $e_1 \neq e \neq e_2$. Therefore $I_\phi(e) = 1$.

(2 $\Rightarrow$ 1). Since $I_\phi(e) = 1$, $\|e\| = 1$. Assume that $e = \frac{1}{2}(e_1 + e_2)$, $\|e_i\| = 1$, $i = 1, 2$. Since I_ϕ and ϕ are convex and $I_\phi(e_i) \leq 1$, we obtain that $I_\phi(e_i) = 1$, $i = 1, 2$, and $\phi(e(w)) = \frac{1}{2}(\phi(e_1(w)) + \phi(e_2(w)))$, $w \in \Omega$, a.e. But ϕ is strictly convex. Therefore $e(w) = e_1(w) = e_2(w)$ a.e. and $e = e_1 = e_2$. $\square$

9. Proposition. Let ϕ be strictly convex and (Ω, Σ, μ) a measure space. Then the following are equivalent:

- (1) $B_{L^\phi(\mu)}$ is stable;
- (2) $E^\phi = L^\phi(\mu)$.

Proof. (2 $\Rightarrow$ 1). This is 3. Proposition.

(1 $\Rightarrow$ 2). First recall that, if $f \in L^\phi(\mu)$ and $I_\phi(f) < 1$, the following are equivalent: (i) $\|f\| = 1$; (ii) $I_\phi(\lambda f) = +\infty$ for each $\lambda > 1$. Suppose that $E^\phi \neq L^\phi(\mu)$. We claim that there exists $f \in S_{L^\phi(\mu)}$ such that $I_\phi(f) < 1$. Indeed, as $E^\phi \neq L^\phi(\mu)$, there exist $g \in S_{L^\phi(\mu)}$ and $\lambda > 1$ such that $I_\phi(\lambda g) = \infty$. Let $a = \inf\{\lambda > 0: I_\phi(\lambda g) = \infty\}$. Clearly $a \geq 1$ because, as $g \in B_{L^\phi(\mu)}$, $I_\phi(g) \leq 1$. If $A = \{w \in \Omega: g(w) \neq 0\}$, take a disjoint partition $A = \bigcup_{n \geq 1} A_n$ such that $\mu(A_n) < \infty$ and g bounded on A_n , $n \geq 1$. Now $\int_{A_n} \phi(\lambda g) d\mu < \infty$ for each $\lambda > 0$ and $n \geq 1$. Since $I_\phi((a - 2^{-k})g) < \infty$ but $I_\phi((a + 2^{-k})g) = \infty$, $k \in \mathbb{N}$, we can choose a sequence $\{N_k\}_{k \geq 1}$ of finite subsets of $\mathbb{N}$ verifying:

- (a) If $k_1 < k_2$, $n < m$ for each $n \in N_{k_1}$, $m \in N_{k_2}$.
- (b) $\sum_{n \in N_k} \int_{A_n} \phi((a - 2^{-k})g) d\mu \leq 2^{-(k+1)}$.
- (c) $\sum_{n \in N_k} \int_{A_n} \phi((a + 2^{-k})g) d\mu \geq k$.

Define $f(w) = (a - 2^{-k})g(w)$ if $w \in A_n$ and $n \in N_k$ and $f(w) = 0$ if $w \notin (\bigcup_{k \geq 1} \bigcup_{n \in N_k} A_n)$. Then $I_\phi(f) < 1$ but $I_\phi(\lambda f) = \infty$ for each $\lambda > 1$. Thus $\|f\| = 1$.

Take this function f and $\varepsilon > 0$ arbitrarily. We prove that there exists $g \in \text{Ext. } B_{L^\phi(\mu)}$ verifying $\|f - g\| \leq \varepsilon$. Since $I_\phi((1 + \varepsilon)f) = +\infty$, there exists $B \in \Sigma$, $0 < \mu(B) < \infty$, such that $\int_B \phi((1 + \varepsilon)f) d\mu \geq 1 - I_\phi(f)$. So there exists $0 < \eta \leq \varepsilon$, verifying

$$\int_{\Omega \setminus B} \phi(f) d\mu + \int_B \phi((1 + \eta)f) d\mu = 1.$$

Define $g(w) = f(w)$ if $w \in \Omega \setminus B$ and $g(w) = (1 + \eta)f(w)$ if $w \in B$. Then, by the above 8. Lemma, $g \in \text{Ext. } B_{L^\phi(\mu)}$ and, as $|f(w) - g(w)| \leq |\varepsilon \cdot f(w)|$,

also $\|f - g\| \leq \varepsilon$. With ε arbitrary, we conclude that f belongs to the closure of $\text{Ext. } B_{L^\phi(\mu)}$. But $\text{Ext. } B_{L^\phi(\mu)}$ is closed because $B_{L^\phi(\mu)}$ is stable. Since clearly $f \notin \text{Ext. } B_{L^\phi(\mu)}$, we obtain a contradiction. $\square$

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